Lecture XII
Power Series Solutions: Ordinary points

1 Analytic function

Definition 1. *Let f be a function defined on an interval $\mathcal{I}$. We say f is analytic at point $x_0 \in \mathcal{I}$ if f can be expanded in a power series about x_0 which has a positive radius of convergence.*

Thus f is analytic at $x = x_0$ if f has the representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n. \tag{1}$$

Here c_n are constant and (1) converges for $|x - x_0| < R$ where $R > 0$. Radius of convergence R can be found from ratio test/root test.
If f has power series representation (1), then its derivative exits in $|x-x_0| < R$. These derivatives are obtained by differentiating the RHS of (1) term by term. Thus,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1} \equiv \sum_{n=0}^{\infty} (n+1) c_{n+1} (x - x_0)^n, \tag{2}$$

and

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x - x_0)^{n-2} \equiv \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} (x - x_0)^n. \tag{3}$$

2 Ordinary points

Consider a linear 2nd order homogeneous ODE of the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0,$$

where a_0, a_1 and a_2 are continuous in an interval $\mathcal{I}$. The points where $a_0(x) = 0$ are called *singular points*. If $a_0(x) \neq 0, \forall x \in \mathcal{I}$, then the above ODE can be written as (by dividing by $a_0(x)$)

$$y'' + p(x)y' + q(x)y = 0. \tag{4}$$

Definition 2. *A point $x_0 \in \mathcal{I}$ is called an ordinary point for (4) if $p(x)$ and $q(x)$ are analytic at $x = x_0$.*

Theorem 1. *Let x_0 be an ordinary point for (4). Then there exists a unique solution $y = y(x)$ of (4) which is also analytic at x_0 and satisfies $y(x_0) = K_0, y'(x_0) = K_1$ (K_0, K_1 are arbitrary constants). Further, if p and q have convergent power series expansion in $|x - x_0| < R$, ($R > 0$), then the power series expansion of y is also convergent in $|x - x_0| < R$.*

Example 1. *Find power series solution around $x_0 = 0$ for*

$$(1 + x^2)y'' + 2xy' - 2y = 0.$$

Solution: $\Big($This can be solved by reduction of order technique since $Y_1 = x$ is a solution. The other solution is given by

$$Y_2(x) = Y_1(x)\int \frac{1}{x^2}e^{-\int 2x/(1+x^2)\,dx}dx = x\int\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)dx = -(1 + x\tan^{-1}x)$$

Thus, two LI solutions are $Y_1 = x$ and $Y_2 = 1 + x\tan^{-1}x\Big)$

Here $p(x) = 2x/(1+x^2)$ and $q(x) = -2/(1+x^2)$ are analytic at $x = 0$ with common radius of convergence $R = 1$. Let

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Now using (3), we get

$$(1+x^2)y''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^n.$$

Note that the summation in the last term can be taken from $n = 0$ since the contributions due to $n = 0$ and $n = 1$ vanish. Thus

$$(1+x^2)y''(x) = \sum_{n=0}^{\infty}\Big[(n+2)(n+1)c_{n+2} + n(n-1)c_n\Big]x^n.$$

Similarly

$$2xy'(x) = \sum_{n=0}^{\infty} 2nc_n x^n.$$

Substuting into the given ODE we find

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)c_{n+2} + n(n-1)c_n + 2nc_n - 2c_n\Big]x^n = 0.$$

Now all the coefficients of powers of x must be zero. Hence,

$$(n+2)(n+1)c_{n+2} = -\Big(n(n-1)c_n + 2nc_n - 2c_n\Big) \Rightarrow c_{n+2} = -\frac{n-1}{n+1}c_n, \qquad n = 0, 1, 2, \cdots.$$

This enables us to find c_n in terms c_0 or c_1. For $n = 0$ get

$$c_2 = c_0,$$

and for $n = 1$ we obtain

$$c_3 = 0.$$

Similarly, letting $n = 2, 3, 4, \cdots$ we find that $c_n = 0$, $n = 5, 7, 9, \cdots$, and

$$c_4 = -\frac{1}{3}c_2 = -\frac{1}{3}c_0, \quad c_6 = -\frac{3}{5}c_4 = \frac{1}{5}c_0, \cdots.$$

By induction we find that for $m = 1, 2, 3, \cdots$,

$$c_{2m} = (-1)^{m-1}\frac{1}{2m-1}c_0,$$

and

$$c_{2m+1} = 0.$$

Now we write

$$y(x) = c_0 y_1(x) + c_1 y_2(x),$$

where

$$y_1(x) = 1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \cdots$$

OR

$$y_1(x) = 1 + x\sum_{m=0}^{\infty}(-1)^m\frac{1}{2m+1}x^{2m+1}$$

and

$$y_2(x) = x.$$

Here c_0 and c_1 are arbitrary. Thus, y_1 is a solution corresponding to $c_0 = 1, c_1 = 0$ and y_2 is a solution corresponding to $c_0 = 0, c_1 = 1$. They form a basis of solutions. Obviously y_2 being polynomial has radius of convergence $R = \infty$ and y_1 has $R = 1$. Thus, the power series solution is valid at least in $|x| < 1$. We can identify y_1 with $1 + x\tan^{-1} x$ obtained earlier.

Comment: In the above problem, it was possible to write the series (after substitution of $y = \sum_{n=0}^{\infty} c_n x^n$) in the form

$$\sum_{n=0}^{\infty} b_n x^n = 0,$$

which ultimately gives $b_n = 0,\ n = 0, 1, 2, \cdots$. Sometimes, we need to leave few terms outside of the summation OR define few new terms inside the summation. For example, consider

$$(1 + x^2)y'' + x^2 y = 0.$$

If we substitute $y = \sum_{n=0}^{\infty} c_n x^n$, then we find

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} c_{n-2}x^n = 0. \quad (5)$$

This can be arranged in two different ways:

(A) Here we write (5) as

$$2c_2 + 3\cdot 2c_3 x + \sum_{n=2}^{\infty}\Big[(n+2)(n+1)c_{n+2} + n(n-1)c_n + c_{n-2}\Big]x^n = 0$$

Hence $c_2 = 0,\ c_3 = 0,\ \ (n+2)(n+1)c_{n+2} + n(n-1)c_n + c_{n-2} = 0,\ n \geq 2$

(B) Here we write (5) as

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)c_{n+2} + n(n-1)c_n + c_{n-2}\Big]x^n = 0, \qquad c_{-2} = c_{-1} = 0.$$

Thus, $(n+2)(n+1)c_{n+2} + n(n-1)c_n + c_{n-2} = 0,\ \ n \geq 0, c_{-2} = c_{-1} = 0$