Lecture XIII

Legendre Equation, Legendre Polynomial

1 Legendre equation

This equation arises in many problems in physics, specially in boundary value problems in spheres:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{1}$$

where α is a constant.

We write this equation as

$$y'' + p(x)y' + q(x)y = 0,$$

where

$$p(x) = \frac{-2x}{1-x^2} \quad \text{and} \quad q(x) = \frac{\alpha(\alpha+1)}{1-x^2}.$$

Clearly $p(x)$ and $q(x)$ are analytic at the origin and have radius of convergence $R = 1$. Hence $x = 0$ is an ordinary point for (1). Assume

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Proceeding as in the case of example 1 in lecture note XII, we find

$$c_{n+2} = -\frac{(\alpha+n+1)(\alpha-n)}{(n+2)(n+1)} c_n, \qquad n = 0, 1, 2, \cdots$$

Taking $n = 0, 1, 2$ and 3 we find

$$c_2 = -\frac{(\alpha+1)\alpha}{1\cdot 2} c_0, \quad c_3 = -\frac{(\alpha+2)(\alpha-1)}{1\cdot 2\cdot 3} c_1, \quad c_4 = \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{1\cdot 2\cdot 3\cdot 4} c_0,$$

and

$$c_5 = \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5} c_1.$$

By induction, we can prove that for $m = 1, 2, 3, \cdots$

$$c_{2m} = (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} c_0$$

$$c_{2m+1} = (-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!} c_1.$$

Thus, we can write

$$y(x) = c_0 y_1(x) + c_1 y_1(x),$$

where

$$y_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} x^{2m}, \tag{2}$$

and

$$y_2(x) = x+\sum_{m=1}^{\infty}(-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}. \tag{3}$$

Taking $c_0 = 1, c_1 = 0$ and $c_0 = 0, c_1 = 0$, we find that y_1 and y_2 are solutions of Legendre equation. Also, these are LI, since their Wronskian is nonzero at $x = 0$. The series expansion for y_1 and y_2 may terminate (in that case the corresponding solution has $R = \infty$), otherwise they have radius of convergence $R = 1$.

2 Legendre polynomial

We note that if α in (1) is a nonnegative integer, then either y_1 given in (2) or y_2 given in (3) terminates. Thus, y_1 terminates when $\alpha = 2m$ $(m = 0, 1, 2, \cdots)$ is nonnegative even integer:

$$\begin{array}{lcll} y_1(x) &=& 1, & (\alpha = 0),\\ y_1(x) &=& 1-3x^2, & (\alpha = 2),\\ y_1(x) &=& 1-10x^2+\frac{35}{3}x^4, & (\alpha = 4). \end{array}$$

Note that y_2 does not terminate when α is a nonnegative even integer.

Similarly, y_2 terminates (but y_1 does not terminate) when $\alpha = 2m+1$ $(m = 0, 1, 2, \cdots)$ is nonnegative odd integer:

$$\begin{array}{lcll} y_2(x) &=& x, & (\alpha = 1),\\ y_2(x) &=& x-\frac{5}{3}x^3, & (\alpha = 5),\\ y_2(x) &=& x-\frac{14}{3}x^2+\frac{21}{5}x^5, & (\alpha = 5). \end{array}$$

Notice that the polynomial solution of

$$(1-x^2)y''-2xy'+n(n+1)y = 0, \tag{4}$$

where n is nonnegative integer, is polynomial of degree n. Equation (4) is the same as (1) with n replacing α.

Definition 1. *The polynomial solution, denoted by $P_n(x)$, of degree n of (4) which satisfies $P_n(1) = 1$ is called the Legendre polynomial of degree n.*

Let ψ be a polynomial of degree n defined by

$$\psi(x) = \frac{d^n}{dx^n}(x^2-1)^n. \tag{5}$$

Then ψ is a solution of (4). To prove it, we proceed as follows: Assume $u(x) = (x^1-1)^n$. Then

$$(x^2-1)u^{(1)} = 2nxu. \tag{6}$$

Now we take $(n+1)$-th derivative of both sides of (6):

$$\Big((x^2-1)u^{(1)}\Big)^{(n+1)} = 2n(xu)^{(n+1)}. \tag{7}$$

Now we use Leibniz rule for the derivative of product two functions f and g:

$$(f \cdot g)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k)},$$

which can be proved easily by induction.
Thus from (7) we get

$$(x^2-1)u^{(n+2)} + 2x(n+1)u^{(n+1)} + (n+1)nu^{(n)} = 2n\Big(xu^{(n+1)} + (n+1)u^{(n)}\Big).$$

Simplifying this and noting that $\psi = u^{(n)}$, we get

$$(1-x^2)\psi'' - 2x\psi' + n(n+1)\psi = 0.$$

Thus, ψ satisfies (4). Note that we can write

$$\psi(x) = \Big((x+1)^n(x-1)^n\Big)^{(n)} = (x+1)^n n! + (x-1)s(x),$$

where $s(x)$ is a polynomial. Thus, $\psi(1) = 2^n n!$. Hence,

$$P_n(x) = \frac{1}{2^n n!}\psi(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{8}$$

3 Properties of Legendre polynomials

a. *Generating function*: The function $G(t, x)$ given by

$$G(t,x) = \frac{1}{\sqrt{1-2xt+t^2}}$$

is called the generating function of the Legendre polynomials. It can be shown that for small t

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

b. *Orthogonality*: The following property holds for Legendre polynomials:

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{2}{2n+1}, & \text{if } m = n. \end{cases}$$

c. *Fourier-Legendre series*: By using the orthogonality of Legendre polynomials, any piecewise continuous function in $-1 \le x \le 1$ can be expresses in terms of Legendre polynomials:

$$f(x) \sim \sum_{n=0}^{\infty} c_n P_n(x),$$

where

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx.$$

Now

$$\sum_{n=0}^{\infty} c_n P_n(x) = \begin{cases} f(x), & \text{where } f \text{ is continuous} \\ \dfrac{f(x^-)+f(x^+)}{2}, & \text{where } f \text{ is discontinuous} \end{cases}$$