Lecture XIV
Frobenius series: Regular singular points

1 Singular points

Consider the second order linear homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \qquad x \in \mathcal{I} \tag{1}$$

Suppose that a_0, a_1 and a_2 are analytic at $x_0 \in \mathcal{I}$. If $a_0(x_0) = 0$, then x_0 is a *singular* point for (1).

Definition 1. *A point $x_0 \in \mathcal{I}$ is a regular singular point for (1) if (1) can be written as*

$$b_0(x)(x-x_0)^2y'' + b_1(x)(x-x_0)y' + b_2(x)y = 0, \tag{2}$$

where $b_0(x_0) \neq 0$ and b_0, b_1, b_2 are analytic at x_0.

Comment 1: Since $b_0(x_0) \neq 0$, we get an equivalent definition of regular singular point by dividing (2) by $b_0(x)$. Thus, a point $x_0 \in \mathcal{I}$ is a regular singular point for (1) if (1) can be written as

$$(x-x_0)^2y'' + (x-x_0)p(x)y' + q(x)y = 0, \tag{3}$$

where p and q are analytic at x_0.
Comment 2: Any singular point of (1) which is not regular is called irregular singular point.

Example 1. *Consider*

$$x^3y'' - (1-\cos x)y' + xy = 0$$

The singular point $x_0 = 0$ is regular.

Example 2. *Consider*

$$x^2(x-1)^2y'' + (\sin x)y' + (x-1)y = 0$$

The singular point $x_0 = 0$ is regular whereas $x_0 = 1$ is irregular.

Example 3. *Euler-Cauchy equation:*

$$ax^2y'' + bxy' + cy = 0,$$

where a, b, c are constants. Here $x_0 = 0$ is a regular singular point.

For simplicity, we consider a second order linear ODE with a regular singular point at $x_0 = 0$. If $x_0 \neq 0$, it is easy to convert the given ODE to an equivalent ODE with regular singular point at $x_0 = 0$. For this, we substitute $t = x - x_0$ and let $z(t) = y(x_0 + t)$. Then (3) becomes

$$t^2\ddot{z} + t\tilde{p}(t)\dot{z} + \tilde{q}(t)z = 0,$$

where $\dot{} = d/dt$. Thus, we consider following second order homogeneous linear ODE

$$x^2y'' + xp(x)y' + q(x)y = 0, \tag{4}$$

where p, q are analytic at the origin.

Ordinary point vs. regular singular point: This can explained by taking two examples. Consider

$$y'' + y = 0,$$

which has 0 as the ordinary point. Note that the general solution is $y = c_1 \cos x + c_2 \sin x$. At the ordinary point $x_0 = 0$, we can find unique c_1, c_2 for a given K_0, K_1 such that $y(0) = K_0, y'(0) = K_1$. Thus, unique solution exists for initial conditions specified at the ordinary point.

Now consider the Euler-Cauchy equation

$$x^2y'' - 2xy' + 2y = 0,$$

for which $x_0 = 0$ is a regular singular point. The general solution is $y = c_1x + c_2x^2$. Now it is not possible to find unique values of c_1, c_2 for a given K_0, K_1 such that $y(0) = K_0, y'(0) = K_1$. Note that solution does not exist for $K_0 \neq 0$ since $y(0) = 0$.

2 Frobenius method

We would like to find two linearly independent solutions of (4) so that these form a basis solution for $x \neq 0$. *We find the basis solution for* $x > 0$. For $x < 0$, we substitute $t = -x$ and carry out similar procedure for $t > 0$.

If p and q in (4) are constants, then a solution of (4) is of the form x^r. But since p and q are power series, we assume that a solution of (4) can be represented by an extended power series

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \tag{5}$$

which is a product of x^r and a power series. We also assume that $a_0 \neq 0$. We formally substitute (5) into (4) and find r and $a_1, a_2, \cdots$ in terms of a_0 and r. Once we find (5), we next check the convergence of the series. If it converges, then (5) becomes solution for (4).

Now from (5), we find

$$x^2y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}, \quad xy'(x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}.$$

Since p and q are analytic, we write

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \quad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Substituting into (4), we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0.$$

OR

$$x^r \sum_{n=0}^{\infty} \left[(r+n)(r+n-1)a_n + \sum_{k=0}^{n} \Big((r+k)p_{n-k} + q_{n-k} \Big) a_k \right] x^n = 0.$$

Since $x > 0$, this becomes

$$\sum_{n=0}^{\infty} \left[(r+n)(r+n-1)a_n + \sum_{k=0}^{n} \Big((r+k)p_{n-k} + q_{n-k} \Big) a_k \right] x^n = 0. \tag{6}$$

Thus, we must have

$$[(r+n)(r+n-1)+(n+r)p_0+q_0]a_n + \sum_{k=0}^{n-1}[(r+k)p_{n-k}+q_{n-k}]a_k = 0, \quad n = 0, 1, 2, \cdots \tag{7}$$

Now (7) gives a_n in terms of $a_0, a_1, \cdots, a_{n-1}$ and r.
For $n = 0$, we find

$$r(r-1) + p_0 r + q_0 = 0, \tag{8}$$

since $a_0 \neq 0$. Equation (8) is called indicial equation for (4). The form of the linearly independent solutions of (4) depends on the roots of (8).
Let $\rho(r) = r(r-1) + p_0 r + q_0$. Then for $n = 1, 2, \cdots$, we find

$$\rho(r+n)a_n + b_n = 0,$$

where

$$b_n = \sum_{k=0}^{n-1}[(r+k)p_{n-k} + q_{n-k}]a_k.$$

Notice that b_n is a linear combination of $a_0, a_1, \cdots, a_{n-1}$. Thus, we can find a_n uniquely in terms of r and a_0 if $\rho(r+n) \neq 0$. If $\rho(r+n) = 0$, then it is possible to find value of a_n in certain cases.
Let r_1, r_2 be the roots of the indicial equation (8). We assume that the roots are real and $r_1 \geq r_2$. For r_1, clearly $\rho(r_1+n) \neq 0$ for $n = 1, 2, \cdots$. Thus, we can determine $a_1, a_2, a_3, \cdots$ corresponding to r_1. Clearly, one Frobenius series (extended power series) solution y_1 corresponding to the larger root r_1 always exists. Suppose $a_0 = 1$, then

$$y_1(x) = x^{r_1}\Big(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\Big). \tag{9}$$

Now for $r = r_2$, three cases may appear. These are as follows:

A. $r_1 - r_2$ **is not a nonnegative integer**: Then $r_2 + n \neq r_1$ for any integer $n \geq 1$ and as a result $\rho(r_2+n) \neq 0$ for any $n \geq 1$.Thus, we can determine $a_1, a_2, a_3, \cdots$ corresponding to r_2. Clearly, another Frobenius series solution y_2 corresponding to the smaller root exists. Suppose $a_0 = 1$, then

$$y_2(x) = x^{r_2}\Big(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\Big). \tag{10}$$

B. $r_1 = r_2$**, double root**: Clearly a second extended power series (Frobenius series) solution does not exist.

C. $r_1 - r_2 = m$, $m \geq 1$ **is a positive integer**: In this case $\rho(r_2 + m) = \rho(r_1) = 0$. Thus, we can find $a_1, a_2, \cdots, a_{m-1}$. But for a_m, we have

$$\rho(r_2 + m)a_m = -b_m.$$

Since $\rho(r) = (r - r_1)(r - r_2)$, we have

$$\rho(r + m) = (r + m - r_1)(r + m - r_2) = (r - r_2)(r + m - r_2).$$

Clearly two cases may arise here:

C.i b_m has a factor $r - r_2$, i.e. $b_m(r_2) = 0$. In this case, we cancel factor $r - r_2$ from both sides and find $a_m(r_2)$ as a finite number. Then we can continue calculating remaining coefficients $a_{m+1}, a_{m+2}, \cdots$. Hence, a second Frobenius series solution exists.

C.ii On the other hand, if $b_m(r_2) \neq 0$, then it is not possible to continue the calculations of a_n for $n \geq m$. Hence, a second Frobenius series solution does not exist.

To find the form of the solution in the case of B and C described above, we use the reduction of order technique. We know that $y_1(x)$ (corresponding the larger root) always exists. Let $y_2(x) = v(x)y_1(x)$. Then

$$\begin{aligned} v' &= \frac{1}{y_1^2} e^{-\int p(x)/x\, dx} \\ &= \frac{1}{x^{2r_1}\Big(1 + a_1(r_1)x + a_2(r_1)x^2 + \cdots\Big)^2} e^{-p_0 \ln x - p_1 x - \cdots} \\ &= \frac{1}{x^{2r_1 + p_0}\Big(1 + a_1(r_1)x + a_2(r_1)x^2 + \cdots\Big)^2} e^{-p_1 x - \cdots} \\ &= \frac{1}{x^{2r_1 + p_0}} g(x), \end{aligned}$$

where $g(x)$ is analytic at $x = 0$ and $g(0) = 1$. Since $g(x)$ is analytic at $x = 0$ with $g(0) = 1$, we must have $g(x) = 1 + \sum_{n=1}^{\infty} g_n x^n$. Since r_1, r_2 are roots of (8), we must have

$$r_1 + r_2 = 1 - p_0 \Rightarrow 2r_1 + p_0 = m + 1.$$

Hence,

$$v' = \frac{1}{x^{m+1}} + \frac{g_1}{x^m} + \cdots + \frac{g_{m-1}}{x^2} + \frac{g_m}{x} + g_{m+1} + \cdots,$$

OR

$$v(x) = \frac{x^{-m}}{-m} + \frac{g_1 x^{-m+1}}{-m+1} + \cdots + \frac{g_{m-1}x^{-1}}{-1} + g_m \ln x + g_{m+1}x + \cdots. \tag{11}$$

Thus,

$$\begin{aligned} y_2(x) &= y_1(x)\left[\frac{x^{-m}}{-m} + \frac{g_1 x^{-m+1}}{-m+1} + \cdots + \frac{g_{m-1}x^{-1}}{-1} + g_m \ln x + g_{m+1}x + \cdots\right] \\ &= g_m y_1(x) \ln x + x^{r_1}\Big(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\Big)\left[\frac{x^{-m}}{-m} + \frac{g_1 x^{-m+1}}{-m+1} + \cdots + \frac{g_{m-1}x^{-1}}{-1} + g_{m+1}x + \cdots\right] \end{aligned}$$

Now we take the factor x^{-m} from the series inside the third bracket. Since $r_1 - m = r_2$, we finally find

$$y_2(x) = cy_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} c_n x^n, \tag{12}$$

where we put $g_m = c$.

Now for $r_1 = r_2$, we have $m = 0$ and hence $g_m = g_0 = g(0) = 1 = c$. Thus, $\ln x$ term is definitely present in the second solution. Also in this case, the series in (11) starts with $g_0 \ln x$ and the next term is $g_1 x$. Hence, for $r_1 = r_2$, we must have $c_0 = 0$ in (12). In certain cases, $g_m = c$ becomes zero (case C.ii) for $m \geq 1$. Then the second solution is also a Frobenius series solution; otherwise, the second Frobenius series solution does not exist.

3 Summary

The results derived in the previous section can be summarized as follows. Consider

$$x^2 y'' + xp(x)y' + q(x)y = 0, \tag{13}$$

where p and q have convergent power series expansion in $|x| < R,\ R > 0$. Let r_1, r_2 ($r_1 \geq r_2$) be the roots of the indicial equation:

$$r^2 + (p(0) - 1)r + q(0) = 0 \tag{14}$$

For $x > 0$ we have the following theorems:

Theorem 1. *If $r_1 - r_2$ is not zero or a positive integer, then there are two linearly independent solutions y_1 and y_2 of (13) of the form*

$$y_1(x) = x^{r_1}\sigma_1(x), \quad y_2(x) = x^{r_2}\sigma_2(x), \tag{15}$$

where σ_1, σ_2 are analytic at $x = 0$ with radius of convergence R and $\sigma_1(0) \neq 0$ and $\sigma_2(0) \neq 0$.

Theorem 2. *If $r_1 = r_2$, then there are two linearly independent solutions y_1 and y_2 of (13) of the form*

$$y_1(x) = x^{r_1}\sigma_1(x), \quad y_2(x) = (\ln x)y_1(x) + x^{r_2+1}\sigma_2(x), \tag{16}$$

where σ_1, σ_2 are analytic at $x = 0$ with radius of convergence R and $\sigma_1(0) \neq 0$.

Theorem 3. *If $r_1 - r_2$ is a positive integer, then there are two linearly independent solutions y_1 and y_2 of (13) of the form*

$$y_1(x) = x^{r_1}\sigma_1(x), \quad y_2(x) = c(\ln x)y_1(x) + x^{r_2}\sigma_2(x), \tag{17}$$

where σ_1, σ_2 are analytic at $x = 0$ with radius of convergence R and $\sigma_1(0) \neq 0$ and $\sigma_2(0) \neq 0$. It may happen that $c = 0$.

Example 4. *Discuss whether two Frobenius series solutions exist or do not exist for the following equations:*
(i) $2x^2y'' + x(x+1)y' - (\cos x)y = 0$,
(ii) $x^4y'' - (x^2 \sin x)y' + 2(1 - \cos x)y = 0$.

Solution: (i) We can write this as

$$x^2y'' + \frac{(x+1)}{2}xy' - \frac{\cos x}{2}y = 0.$$

Hence $p(x) = (x+1)/2$ and $q(x) = -\cos x/2$. Thus, $p(0) = 1/2$ and $q(0) = -1/2$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow 2r^2 - r - 1 = 0 \Rightarrow r_1 = 1, r_2 = -1/2.$$

Since $r_1 - r_2 = 3/2$, which is not zero or a positive integer, two Frobenius series solutions exist.
(ii) We can write this as

$$x^2y'' - \frac{\sin x}{x}xy' + 2\frac{1-\cos x}{x^2}y = 0.$$

Hence $p(x) = -\sin x/x$ and $q(x) = 2(1 - \cos x)/x^2$. Thus, $p(0) = -1$ and $q(0) = 1$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow r_1 = 1 = r_2.$$

Since $r_1 = r_2$, only one Frobenius series solutions exists.

Example 5. (Case A) *Find two independent solutions around* $x = 0$ *for*

$$2xy'' + (x+1)y' + 3y = 0$$

Solution: We write this as

$$x^2y'' + \frac{(x+1)}{2}xy' + (3x/2)y = 0.$$

Hence $p(x) = (x+1)/2$ and $q(x) = 3x/2$. Thus, $p(0) = 1/2, q(0) = 0$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow 2r^2 - r = 0 \Rightarrow r_1 = 1/2, r_2 = 0.$$

Since $r_1 - r_2 = 1/2$, is not zero or a positive integer, two independent Frobenius series solution exist.
Substituting

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

(after some manipulation and cancelling x^r) we find

$$\sum_{n=0}^{\infty} \rho(n+r)a_n x^n + \sum_{n=1}^{\infty} \Big((n+r-1) + 3\Big)a_{n-1}x^n = 0,$$

where $\rho(r) = r(2r-1)$. Rearranging the above, we get

$$\rho(r)a_0 + \sum_{n=1}^{\infty}\Big[\rho(n+r)a_n + (n+r+2)a_{n-1}\Big]x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \quad \rho(n+r)a_n + (n+r+2)a_{n-1} = 0 \ \text{ for } n \geq 1.$$

From the first relation we find roots of the indicial equation $r_1 = 1/2, r_2 = 0$. Now with the larger root $r = r_1 = 1/2$, we find

$$a_n = -\frac{(2n+5)a_{n-1}}{2n(2n+1)}, \quad n \geq 1.$$

Iterating we find

$$a_1 = -\frac{7}{6}a_0, \quad a_2 = \frac{21}{40}a_0, \cdots$$

Hence, by induction

$$a_n = (-1)^n \frac{(2n+5)(2n+3)}{15 \cdot 2^n n!}a_0, \quad n \geq 1 \qquad \text{(Check!)}$$

Thus, taking $a_0 = 1$, we find

$$y_1(x) = x^{1/2}\left(1 - \frac{7}{6}x + \frac{21}{40}x^2 - \cdots\right)$$

Now with $r = r_2 = 0$, we find

$$a_n = -\frac{(n+2)a_{n-1}}{n(2n-1)}, \quad n \geq 1.$$

Iterating we find

$$a_1 = -3a_0, \quad a_2 = 2a_0, \cdots$$

Hence, by induction

$$a_n = (-1)^n\left(\frac{5}{2n-1} - \frac{2}{n}\right)\left(\frac{5}{2n-3} - \frac{2}{n-1}\right)\cdots\left(\frac{5}{1} - \frac{2}{1}\right)a_0, \quad n \geq 1 \qquad \text{(Check!)}$$

Thus, taking $a_0 = 1$, we find

$$y_2(x) = \left(1 - 3x + 2x^2 - \cdots\right)$$

Example 6. (Case B) *Find the general solution in the neighbourhood of origin for*

$$4x^2y'' - 8x^2y' + (4x^2+1)y = 0$$

Solution: We write this as

$$x^2y'' - (2x)xy' + (x^2 + 1/4)y = 0.$$

Hence $p(x) = -2x$ and $q(x) = x^2 + 1/4$. Thus, $p(0) = 0, q(0) = 1/4$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow r^2 - r + 1/4 = 0 \Rightarrow r_1 = r_2 = 1/2.$$

Since the indicial equation has a double root, only one Frobenius series solution exists. Substituting

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

(after some manipulation and cancelling x^r) we find

$$\sum_{n=0}^{\infty} \rho(n+r)a_n x^n - \sum_{n=1}^{\infty} 8(n+r-1)a_{n-1}x^n + \sum_{n=2}^{\infty} 4a_{n-2}x^n = 0,$$

where $\rho(r) = (2r-1)^2$. Rearranging the above, we get

$$\rho(r)a_0 + \Big(\rho(r+1)a_1 - 8ra_0\Big)x + \sum_{n=2}^{\infty} \Big[\rho(n+r)a_n - 8(n+r-1)a_{n-1} + 4a_{n-2}\Big]x^n = 0.$$

Now with $r = 1/2$, we find

$$a_1 = a_0,\ a_n = \frac{(2n-1)a_{n-1}}{n^2} - \frac{a_{n-2}}{n^2}, \quad n \geq 2.$$

Iterating we find

$$a_2 = \frac{1}{2!}a_0, \quad a_3 = \frac{1}{3!}a_0, \quad a_4 = \frac{1}{4!}a_0, \cdots$$

Hence, by induction

$$a_n = \frac{1}{n!}a_0, \quad n \geq 1.$$

$\{$Induction: Claim $a_k = a_0/k!$. True for $k = 1, 2$. Assume it is true for $k = m$. Now for $k = m + 1$,

$$a_{k+1} = \frac{(2k+1)a_k}{(k+1)^2}a_0 - \frac{a_{k-1}}{(k+1)^2}a_0 = \frac{1}{(k-1)!(k+1)^2}\frac{k+1}{k}a_0 = \frac{a_0}{(k+1)!} \quad \Big\}$$

Thus, taking $a_0 = 1$, we find

$$y_1(x) = x^{1/2}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = x^{1/2}e^x.$$

For the general solution, we need to find another solution y_2. For this we use reduction of order. Let $y_2(x) = y_1(x)v(x)$. Then

$$v = \int \frac{1}{y_1^2}e^{-\int p dx}\, dx,$$

where $p(x) = -2$. Hence

$$v(x) = \int \frac{1}{x}dx = \ln x$$

and $y_2 = (\ln x)x^{1/2}e^x$. Thus, the general solution is

$$y(x) = x^{1/2}e^x(c_1 + c_2 \ln x)$$

Example 7. (Case C.i) *Find two independent solutions around $x = 0$ for*

$$xy'' + 2y' + xy = 0$$

Solution: We write this as

$$x^2y'' + 2xy' + x^2y = 0.$$

Hence $p(x) = 2$ and $q(x) = x^2$. Thus, $p(0) = 2, q(0) = 0$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow r^2 + r = 0 \Rightarrow r_1 = 0, r_2 = -1.$$

A Frobenius series solution exists for the larger root $r_1 = 0$. Substituting

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

(after some manipulation and cancelling x^r) we find

$$\sum_{n=0}^{\infty} \rho(n+r)a_n x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0,$$

where $\rho(r) = r(r+1)$. Rearranging the above, we get

$$\rho(r)a_0 + \rho(r+1)a_1 x + \sum_{n=2}^{\infty} \Big[\rho(n+r)a_n + a_{n-2}\Big] x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \quad \rho(r+1)a_1 = 0, \quad \rho(n+r)a_n + a_{n-2} = 0 \quad \text{for } n \geq 2.$$

From the first relation we find roots of the indicial equation $r_1 = 0, r_2 = -1$. Now with the larger root $r = r_1$, we find

$$a_1 = 0, a_n = -\frac{a_{n-2}}{n(n+1)}, \quad n \geq 2.$$

Iterating we find

$$a_2 = -\frac{1}{3!}a_0, \quad a_3 = 0, \quad a_4 = \frac{1}{5!}a_0, \cdots$$

Hence, by induction

$$a_{2n} = (-1)^n \frac{1}{(2n+1)!}a_0, \quad a_{2n+1} = 0.$$

Thus, taking $a_0 = 1$, we find

$$y_1(x) = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = \frac{\sin x}{x}$$

Since $r_1 - r_2 = 1$, a positive integer, the second Frobenius series solution may or may not exist. Hence, to be sure, we need to compute it. With $r = r_2 = -1$, we find

$$0 \cdot a_1 = 0, \; a_n = -\frac{a_{n-2}}{n(n-1)}, \quad n \geq 2.$$

Now the first relation can be satisfied by taking any value of a_1. For simplicity, we choose $a_1 = 0$. Iterating we find

$$a_2 = -\frac{1}{2!}a_0, \quad a_3 = 0, \quad a_4 = \frac{1}{4!}a_0, \cdots$$

Hence, by induction

$$a_{2n} = (-1)^n \frac{1}{(2n)!}a_0, \quad a_{2n+1} = 0.$$

Thus, indeed a second Frobenius series solution exists and taking $a_0 = 1$, we get

$$y_2(x) = x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = \frac{\cos x}{x}.$$

Comment: The second solution could have been obtained using reduction of order also. Suppose $y_2 = vy_1$, then

$$v = \int \frac{x^2}{\sin^2 x} e^{-\int 2/x\, dx} dx = \int \operatorname{cosec}^2 x\, dx = -\cot x.$$

Hence $y_2(x) = \cos x/x$ (disregarding minus sign)

Example 8. (Case C.ii) *Find general solution around $x = 0$ for*

$$(x^2 - x)y'' - xy' + y = 0$$

Solution: We write this as

$$x^2 y'' - \frac{x}{x-1}xy' + \frac{x}{x-1}y = 0.$$

Hence $p(x) = -x/(x-1)$ and $q(x) = x/(x-1)$. Thus, $p(0) = q(0) = 0$. The indicial equation is

$$r^2 + (p(0) - 1)r + q(0) = 0 \Rightarrow r^2 - r = 0 \Rightarrow r_1 = 1, r_2 = 0.$$

Since $r_1 - r_2 = 1$, a positive integer, two independent Frobenius series solution may or may not exist.
Substituting

$$y = x^r \sum_{n=0}^{\infty} a_n x^n,$$

(after some manipulation and cancelling x^r) we find

$$(x-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^n - x\sum_{n=0}^{\infty}(n+r)a_n x^n + x\sum_{n=0}^{\infty} a_n x^n = 0.$$

Rearranging the above, we get

$$(x-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^n - x\sum_{n=0}^{\infty}\Big((n+r)-1\Big)a_n x^n = 0.$$

OR

$$x \sum_{n=0}^{\infty} (n+r-1)^2 a_n x^n - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^n = 0.$$

OR

$$\sum_{n=1}^{\infty} (n+r-2)^2 a_{n-1} x^n - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^n = 0.$$

OR

$$r(r-1)a_0 + \sum_{n=1}^{\infty} \Big[(n+r)(n+r-1)a_n - (n+r-2)^2 a_{n-1} \Big] x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0, \quad \rho(n+r)a_n - (n+r-2)^2 a_{n-1} = 0 \text{for } n \geq 1,$$

where $\rho(r) = r(r-1)$. From the first relation we find roots of the indicial equation $r_1 = 1, r_2 = 0$. Now with the larger root $r = r_1 = 1$, we find

$$a_n = \frac{(n-1)a_{n-1}}{n(n+1)}, \quad n \geq 1.$$

Iterating we find

$$a_n = 0, \qquad n \geq 1.$$

Thus, taking $a_0 = 1$, we find

$$y_1(x) = x$$

Now with $r = r_2 = 0$, we find

$$n(n-1)a_n = (n-2)^2 a_{n-1}, \quad n \geq 1.$$

Now for $n = 1$, we find $0 = a_0$ which is a contradiction. Hence, second Frobenius series solution does not exist. To find the second independent solution, we use reduction of order technique. Let $y_2(x) = v(x)y_1(x)$. Then

$$v(x) = \int \frac{1}{y_1^2} e^{-\int p dx} dx,$$

where $p(x) = -x/(x^2 - x) = -1/(x-1)$. Hence,

$$v(x) = \int \frac{1}{x^2} e^{\ln(1-x)} \, dx = \int \left(\frac{1}{x^2} - \frac{1}{x} \right) dx = -\left(\frac{1}{x} + \ln x \right).$$

(Why I wrote $\ln(1-x)$ NOT $\ln(x-1)$?) Hence, $y_2(x) = (1 + x \ln x)$ (disregarding the minus sign, since the ODE is homogeneous and linear). Thus, the general solution is given by

$$y(x) = c_1 x + c_2 (1 + x \ln x).$$