Lecture XV

Bessel's equation, Bessel's function

1 Gamma function

Gamma function is defined by

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1}\, dt, \qquad p > 0. \tag{1}$$

The integral in (1) is convergent that can be proved easily. Some special properties of the gamma function are the following:

i. It is readily seen that $\Gamma(p+1) = p\Gamma(p)$, since

$$\begin{aligned} \Gamma(p+1) &= \lim_{T\to\infty} \int_0^T e^{-t} t^p\, dt \\ &= \lim_{T\to\infty} \left[-e^{-t} t^p \Big|_0^T + p \int_0^T e^{-t} t^{p-1}\, dt \right] \\ &= p \int_0^\infty e^{-t} t^{p-1}\, dt = p\Gamma(p). \end{aligned}$$

ii. $\Gamma(1) = 1$ (trivial proof)

iii. If $p = m$, a positive integer, then $\Gamma(m+1) = m!$ (use i. repeatedly)

iv. $\Gamma(1/2) = \sqrt{\pi}$. This can be proved as follows:

$$I = \Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2}\, dt = 2\int_0^\infty e^{-u^2}\, du.$$

Hence

$$I^2 = 4\int_0^\infty e^{-u^2}\, du \int_0^\infty e^{-v^2}\, dv = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy.$$

Using polar coordinates ρ, θ, the above becomes

$$I^2 = 4\int_0^\infty \int_0^{\pi/2} e^{-\rho^2} \rho\, d\rho\, d\theta \Rightarrow I^2 = \pi \Rightarrow I = \sqrt{\pi}$$

v. Using relation in i., we can extend the definition of $\Gamma(p)$ for $p < 0$. Suppose N is a positive integer and $-N < p < -N+1$. Now using relation of i., we find

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} = \frac{\Gamma(p+2)}{p(p+1)} = \cdots = \frac{\Gamma(p+N)}{p(p+1)\cdots(p+N-1)}.$$

Since $p + N > 0$, the above relation is well defined.

vi. $\Gamma(p)$ is not defined when p is zero or a negative integer. For small positive ϵ,

$$\Gamma(\pm\epsilon) = \frac{\Gamma(1\pm\epsilon)}{\pm\epsilon} \approx \frac{1}{\pm\epsilon} \to \pm\infty \quad \text{as} \quad \epsilon \to 0.$$

Since $\Gamma(0)$ is undefined, $\Gamma(p)$ is also undefined when p is a negative integer.

2 Bessel's equation

Bessel's equation of order ν ($\nu \geq 0$) is given by

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \tag{2}$$

Obviously, $x = 0$ is regular singular point. Since $p(0) = 1, q(0) = -\nu^2$, the indicial equation is given by

$$r^2 - \nu^2 = 0.$$

Hence, $r_1 = \nu, r_2 = -\nu$ and $r_1 - r_2 = 2\nu$. A Frobenius series solution exists for the larger root $r = r_1 = \nu$. To find this series, we substitute

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, \qquad x > 0$$

into (2) and (after some manipulation) find

$$\sum_{n=0}^{\infty} \rho(n+r) a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

where $\rho(r) = r^2 - \nu^2$. This equation is rearranged as

$$\rho(r)a_0 + \rho(r+1)a_1 x + \sum_{n=2}^{\infty} \Big(\rho(n+r)a_n + a_{n-2}\Big)x^n = 0.$$

Hence, we find (since $a_0 \neq 0$)

$$\rho(r) = 0,\ \rho(r+1)a_1 = 0,\ \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.$$

From the first relation, we get $r_1 = \nu, r_2 = -\nu$. Now with the larger root $r = r_1$ we find

$$a_1 = 0, \quad a_n = -\frac{a_{n-2}}{n(n+2\nu)}, \quad n \geq 2.$$

Iterating we find (by induction),

$$a_{2n+1} = 0, \quad a_{2n} = (-1)^n \frac{1}{2^{2n} n! (\nu+1)(\nu+2)\cdots(\nu+n)} a_0, \quad n \geq 1.$$

Hence

$$y_1(x) = a_0 x^{\nu} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! (\nu+1)(\nu+2)\cdots(\nu+n)}\right). \tag{3}$$

Here it is usual to choose (*instead of* $a_0 = 1$ *as was done in lecture 14*)

$$a_0 = \frac{1}{2^{\nu}\Gamma(\nu+1)}.$$

Then the Frobenius series solution (3) is called the Bessel function of order ν of the first kind and is denoted by $J_\nu(x)$:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{4}$$

To find the second independent solution, we consider the following three cases:

A. $r_1 - r_2 = 2\nu$ **is not a nonnegative integer:** We know that a second Frobenius series solution for $r_2 = -\nu$ exist. We do similar calculation as in the case of r_1 and it turns out that the resulting series is given by (4) with ν replaced by $-\nu$. Hence, the second solution is given by

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}. \tag{5}$$

B. $r_1 = r_2$**:** Obviously this corresponds to $\nu = 0$ and a second Frobenius series solution does not exist.

C. $r_1 - r_2 = 2\nu$ **is a positive integer:** Now there are two cases. We discuss them separately.

C.i ν **is not a positive integer:** Clearly $\nu = (2k+1)/2$, where $k \in \{0, 1, 2, \cdots\}$. Now we have found earlier that (since $a_0 \neq 0$)

$$\rho(r) = 0,\ \rho(r+1)a_1 = 0,\ \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.$$

With $r = r_2 = -\nu$, we get

$$\rho(r) = 0;\ 1 \cdot \Big(1 - (2k+1)\Big)a_1 = 0;\ n \cdot \Big(n - (2k+1)\Big)a_n = -a_{n-2}, \quad n \geq 2.$$

It is clear that the even terms a_{2n} can be determined uniquely. For odd terms, $a_1 = a_3 = \cdots = a_{2k-1} = 0$ but for a_{2k+1} we must have

$$n \cdot 0 \cdot a_{2k+1} = -a_{2k-1} \Rightarrow 0 \cdot a_{2k+1} = 0.$$

This can be satisfied by taking any value of a_{2k+1} and for simplicity, we can take $a_{2k+1} = 0$. Rest of the odd terms thus also vanish. Hence, the second solution in this case is also given by (5), i.e.

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}. \tag{6}$$

C.ii ν **is a positive integer:** Clearly $\nu = k$, where $k \in \{1, 2, 3, \cdots\}$. Now we find (since $a_0 \neq 0$)

$$\rho(r) = 0,\ \rho(r+1)a_1 = 0,\ \rho(r+n)a_n = -a_{n-2}, \quad n \geq 2.$$

With $r = r_2 = -\nu$, we get

$$\rho(r) = 0;\ 1 \cdot \Big(1 - 2k\Big)a_1 = 0;\ n \cdot \Big(n - 2k\Big)a_n = -a_{n-2}, \quad n \geq 2.$$

It is clear that all the odd terms a_{2n+1} vanish. For even terms, $a_2, a_4, \cdots, a_{2k-2}$ each is nonzero. For a_{2k} we must have

$$n \cdot 0 \cdot a_{2k} = -a_{2k-2} \Rightarrow 0 \cdot a_{2k} \neq 0,$$

which is a contradiction. Thus, a second Frobenius series solution does not exist in this case.

Summary of solutions for Bessel's equation: The Bessel's equation of order ν ($\nu \geq 0$)

$$x^2y'' + xy' + (x^2 - \nu^)y = 0,$$

has two independent Frobenius series solutions J_ν and $J_{-\nu}$ when ν is not an (nonnegative) integer:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}, \qquad J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n-\nu}.$$

Thus the general solution, when ν is not an (nonnegative) integer, is

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x).$$

When ν is a (nonnegative) integer, a second solution, which is independent of J_ν, can be found. This solution is called Bessel function of second kind and is denoted by Y_ν. Hence, the general solution, when ν is an (nonnegative) integer, is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x).$$

3 Linear dependence of J_m and J_{-m}, m is a +ve integer

When $\nu = m$ is a positive integer, then

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2n+m} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{2n+m},$$

since $\Gamma(n+m+1) = (n+m)!$.
Since $\Gamma(\pm 0) = \pm\infty$, we define $1/\Gamma(k)$ to be zero when k is nonpositive integer. Now

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n-m}.$$

Now each term in the sum corresponding to $n = 0$ to $n = m-1$ is zero since $1/\Gamma(k)$ is zero when k is nonpositive integer. Hence, we write the sum starting from $n = m$:

$$J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n-m}.$$

Substituting $n - m = k$, we find

$$\begin{aligned} J_{-m}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^{k+m}}{(m+k)!\Gamma(k+1)} \left(\frac{x}{2}\right)^{2(m+k)-m} \\ &= (-1)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m} \\ &= (-1)^m J_m(x). \end{aligned}$$

Hence J_m and J_{-m} becomes linearly dependent when m is a positive integer.

4 Properties of Bessel function

Few important relationships are very useful in application. These are described here.

A. From the expression for J_ν given in (4), we find

$$x^\nu J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+2\nu}$$

Taking derivative with respect to x we find

$$\Big(x^\nu J_\nu(x)\Big)' = \sum_{n=0}^{\infty} \frac{(-1)^n (n+\nu)}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+2\nu-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu)} \left(\frac{x}{2}\right)^{2n+2\nu-1},$$

where we have used $\Gamma(n+\nu+1) = (n+\nu)\Gamma(n+\nu)$. We can write the above relation as

$$\Big(x^\nu J_\nu(x)\Big)' = x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\Big(n+(\nu-1)+1\Big)} \left(\frac{x}{2}\right)^{2n+\nu-1}.$$

Hence,

$$\Big(x^\nu J_\nu(x)\Big)' = x^\nu J_{\nu-1}(x). \tag{7}$$

B. From (4), we find

$$x^{-\nu} J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(n+\nu+1)} x^{2n}.$$

Taking derivative with respect to x we find

$$\Big(x^{-\nu} J_\nu(x)\Big)' = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+\nu-1} (n-1)!\Gamma(n+\nu+1)} x^{2n-1}.$$

Note that the sum runs from $n = 1$ (in contrast to that in A). Let $k = n-1$, then we obtain

$$\begin{aligned} \Big(x^{-\nu} J_\nu(x)\Big)' &= x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!\Gamma\Big(k+(\nu+1)+1\Big)} \left(\frac{x}{2}\right)^{2k+\nu+1} \\ &= -x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!\Gamma\Big(k+(\nu+1)+1\Big)} \left(\frac{x}{2}\right)^{2k+\nu+1}. \end{aligned}$$

Hence,

$$\Big(x^{-\nu} J_\nu(x)\Big)' = -x^{-\nu} J_{\nu+1}(x). \tag{8}$$

Note: In the first relation A, while taking derivative, we keep the sum running from $n = 0$. This is true only when $\nu > 0$. In the second relation B, we only need $\nu \geq 0$. Taking $\nu = 0$ in B, we find $J_0' = -J_1$. If we put $\nu = 0$ in A, then we find $J_0' = J_{-1}$. But $J_{-1} = -J_1$ and hence we find the same relation as that in B. Hence, the first relation is also valid for $\nu \geq 0$.

C. From A and B, we get

$$\begin{aligned} J'_\nu + \frac{\nu}{x} J_\nu &= J_{\nu-1} \\ J'_\nu - \frac{\nu}{x} J_\nu &= -J_{\nu+1} \end{aligned}$$

Adding and subtracting we find

$$J_{\nu-1} - J_{\nu+1} = 2J'_\nu \tag{9}$$

and

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu. \tag{10}$$