Lecture XVII

Laplace Transform, inverse Laplace Transform, Existence and Properties of Laplace Transform

1 Introduction

Differential equations, whether ordinary or partial, describe the ways certain quantities of interest vary over time. These equations are generally coupled with initial conditions at time $t = 0$ and boundary conditions.

Laplace transform is a powerful technique to solve differential equations. It transforms an IVP in ODE to algebraic equations. The solution of the algebraic equations is than back-transformed to the original problem. In case of PDE, it can be applied to any independent variable x, y, z and t that varies from 0 to ∞. After applying Laplace transform, the original PDE becomes a new PDE with one less independent variable or an ODE. The resulting problem can be solved by other method (such as separation of variables, another transform) and the solution is again back-transformed to the original problem.

2 Laplace transform

Let a function f be defined for $t \geq 0$. We define the Laplace transform of f, denoted by $F(s)$ or $\mathcal{L}\Big(f(t)\Big)$, as

$$F(s) = \mathcal{L}\Big(f(t)\Big) = \int_0^\infty e^{-st} f(t)\, dt, \tag{1}$$

for those s for which the integral in (1) exists. We also refer $f(t)$ as the inverse Laplace transform of $F(s)$ and we write

$$f(t) = \mathcal{L}^{-1}\Big(F(s)\Big).$$

Comment 1: Laplace transform is defined for complex valued function $f(t)$ and the parameter s can also be complex. But we restrict our discussion only for the case in which $f(t)$ is real valued and s is real.

Comment 2: Since the integral in (1) is an improper integral, existence of Laplace transform implies that the following limit exists:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt.$$

Example 1. *Consider the function defined by*

$$f(t) = \begin{cases} 0, & 0 \leq t < a, \\ 1, & t > a. \end{cases}$$

Now

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_a^\infty e^{-st}\,dt \\ &= \frac{e^{-as}}{s}, \qquad\qquad s > 0. \end{aligned}$$

Example 2. *Consider* $f(t) = t^a,\ a > -1$. *Now*

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} t^a\,dt \\ &= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^{a+1-1}\,du \\ &= \frac{\Gamma(a+1)}{s^{a+1}}, \qquad\qquad s > 0. \end{aligned}$$

Hence $F(1) = 1/s;\ F(t) = 1/s^2;\ F(t^n) = n!/s^{n+1}$, *where* n *is nonnegative integer.*

Example 3. *Consider* $f(t) = e^{at}$. *Now*

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} e^{at}\,dt \\ &= \int_0^\infty e^{-(s-a)t}\,dt \\ &= \frac{1}{s-a}, \qquad\qquad s > a. \end{aligned}$$

Example 4. *Consider* $f(t) = \cos(\omega t)$. *Using*

$$\int \cos(at) e^{bt}\,dt = \frac{e^{bt}}{a^2+b^2}\Big(b\cos(at) + a\sin(at)\Big)$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \cos(\omega t)\,dt \\ &= \frac{s}{s^2+\omega^2}, \qquad\qquad s > 0. \end{aligned}$$

Example 5. *Consider* $f(t) = \sin(\omega t)$. *Using*

$$\int \sin(at) e^{bt}\,dt = \frac{e^{bt}}{a^2+b^2}\Big(b\sin(at) - a\cos(at)\Big)$$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \sin(\omega t)\,dt \\ &= \frac{\omega}{s^2+\omega^2}, \qquad\qquad s > 0. \end{aligned}$$

2.1 Existence of Laplace transform

We give sufficient condition for the existence of LT. We need the concept of piecewise continuous function.

Definition 1. (**Piecewise continuous function**) *A function f is piecewise continuous on the interval $[a, b]$ if*
(i) The interval $[a, b]$ can be broken into a finite number of subintervals $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, such that f is continuous in each subinterval (t_i, t_{i+1}), for $i = 0, 1, 2, \cdots, n-1$
(ii) The function f has jump discontinuity at t_i, thus

$$\left|\lim_{t\to t_i^+} f(t)\right| < \infty, \quad i = 0, 1, 2, \cdots, n-1; \qquad \left|\lim_{t\to t_i^-} f(t)\right| < \infty, \quad i = 1, 2, 3, \cdots, n.$$

Note: A function is piecewise continuous on $[0, \infty)$ if it is piecewise continuous in $[0, A]$ for all $A > 0$

Example 6. *The function defined by*

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ 3-t, & 1 < t \le 2, \\ t+1, & 2 < t \le 3, \end{cases}$$

is piecewise continuous on $[0, 3]$

Example 7. *The function defined by*

$$f(t) = \begin{cases} \dfrac{1}{2-t}, & 0 \le t < 2, \\ t+1, & 2 \le t \le 3, \end{cases}$$

is NOT piecewise continuous on $[0, 3]$

Definition 2. (**Exponential order**) *A function f is said to be of exponential order if there exist constants M and c such that*

$$|f(t)| \le Me^{ct} \qquad \text{for sufficiently large } t.$$

Example 8. *Any polynomial is of exponential order. This is clear from the fact that*

$$e^{at} = \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} \ge \frac{t^n a^n}{n!} \implies t^n \le \frac{n!}{a^n} e^{at}$$

But $f(t) = e^{t^2}$ is not of exponential order.

Sufficient condition for the existence of Laplace transform: Let f be a piecewise continuous function in $[0, \infty)$ and is of exponential order. Then Laplace transform $F(s)$ of f exists for $s > c$, where c is a real number that depends on f.

Proof: Since f is of exponential order, there exists A, M, c such that

$$|f(t)| \le Me^{ct} \qquad \text{for } t \ge A.$$

Now we write

$$I = \int_0^\infty f(t)e^{-st}\,dt = I_1 + I_2,$$

where

$$I_1 = \int_0^A f(t)e^{-st}\,dt \quad \text{and} \quad I_2 = \int_A^\infty f(t)e^{-st}.$$

Since f is piecewise continuous, I_1 exists. For the second integral I_2, we note that for $t \ge A$

$$|e^{-st}f(t)| \le Me^{-(s-c)t}.$$

Thus

$$\int_A^\infty |f(t)e^{-st}|\,dt \le M\int_A^\infty e^{-(s-c)t}\,dt \le M\int_0^\infty e^{-(s-c)t}\,dt = \frac{M}{s-c}, \quad s > c.$$

Since the integral in I_2 converges absolutely for $s > c$, I_2 converges for $s > c$. Thus, both I_1 and I_2 exist and hence I exists for $s > c$.

Comment The above condition is not necessary. For example, consider $f(t) = 1/\sqrt{t}$, which is not piecewise continuous in $[0, \infty)$. But

$$\int_0^\infty \frac{e^{-st}}{\sqrt{t}} = \frac{1}{\sqrt{s}}\int_0^\infty e^{-u}u^{1/2-1}\,du = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, \quad s > 0.$$

3 Basic properties of Laplace transform

Theorem 1. *(***Uniqueness of Laplace transform***) Let $f(t)$ and $g(t)$ be two functions such that $F(s) = G(s)$ for all $s > k$. Then $f(t) = g(t)$ at all t where both are continuous.*

Proposition 1. *(***Linearity***) Suppose $F_1(s) = \mathcal{L}\Big(f_1(t)\Big)$ exists for $s > a_1$ and $F_2(s) = \mathcal{L}\Big(f_2(t)\Big)$ exists for $s > a_2$. Then*

$$\mathcal{L}\Big(c_1f_1(t) + c_2f_2(t)\Big) = c_1F_1(s) + c_2F_2(s),$$

for $s > a$, where $a = max\{a_1, a_2\}$.

Proof: Trivial

Example 9. *Consider $f(t) = \cosh(\omega t)$. Then*

$$\mathcal{L}\Big(\cosh(\omega t)\Big) = \frac{1}{2}\Big(\mathcal{L}(e^{\omega t}) + \mathcal{L}(e^{-\omega t})\Big) = \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2-\omega^2}.$$

Example 10. *Consider $f(t) = \sinh(\omega t)$. Proceeding as above, we find*

$$F(s) = \omega/(s^2 - \omega^2)$$

Theorem 2. *(**First shifting theorem**) If $\mathcal{L}\Big(f(t)\Big) = F(s)$, then*

$$\mathcal{L}\Big(e^{at}f(t)\Big) = F(s-a), \quad \textit{and} \quad e^{at}f(t) = \mathcal{L}^{-1}\Big(F(s-a)\Big).$$

Proof: Suppose $\mathcal{L}\Big(f(t)\Big) = F(s)$ holds for $s > k$. Now

$$\mathcal{L}\Big(e^{at}f(t)\Big) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a), \qquad s-a > k.$$

Example 11. *Consider $f(t) = e^{-5t}\cos(4t)$. Since*

$$\mathcal{L}\Big(\cos(4t)\Big) = \frac{s}{s^2+16} \implies \mathcal{L}\Big(e^{-5t}\cos(4t)\Big) = \frac{s+5}{(s+5)^2+16}$$

Proposition 2. *If $\mathcal{L}\Big(f(t)\Big) = F(s)$, then $F(s) \to 0$ as $s \to \infty$.*

Proof: We prove this for a piecewise continuous function which is of exponential order. But the result is valid for any function for which Laplace transform exists. Now

$$I = \int_0^\infty e^{-st}f(t)\,dt \implies |I| \le \int_0^\infty e^{-st}|f(t)|\,dt.$$

Now since the function is exponential order, there exists M, α, A such that $|f(t)| \le M_1 e^{\alpha t}$ for $t \ge A$. Also, since the function is piecewise continuous in $[0, A]$, we must have $|f(t)| \le M_2 e^{\beta t}$ for $0 \le t \le A$ except possibly at some finite number of points where $f(t)$ is not defined. Now we take $M = \max\{M_1, M_2\}$ and $\gamma = \max\{\alpha, \beta\}$. Then we have

$$|F(s)| = |I| \le \int_0^\infty e^{-st}|f(t)|\,dt \le M\int_0^\infty e^{-(s-\gamma)t}\,dt = \frac{M}{s-\gamma}, \quad s > \gamma.$$

Thus, $F(s) \to 0$ as $s \to \infty$.
Comment: Any function $F(s)$ without this behaviour can not be Laplace transform of a certain function. For example, $s/(s-1), \sin s, s^2/(1+s^2)$ are not Laplace transform of any function.