Lecture XVIII

Unit step function, Laplace Transform of Derivatives and Integration, Derivative and Integration of Laplace Transforms

# 1 Unit step function $u_a(t)$

**Definition 1.** *The unit step function (or Heaviside function) $u_a(t)$ is defined*

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a. \end{cases}$$

This function acts as a mathematical 'on-off' switch as can be seen from the Figure 1. It has been shown in Example 1 of Lecture Note 17 that for $a > 0$, $\mathcal{L}\Big(u_a(t)\Big) = e^{-as}/s$.

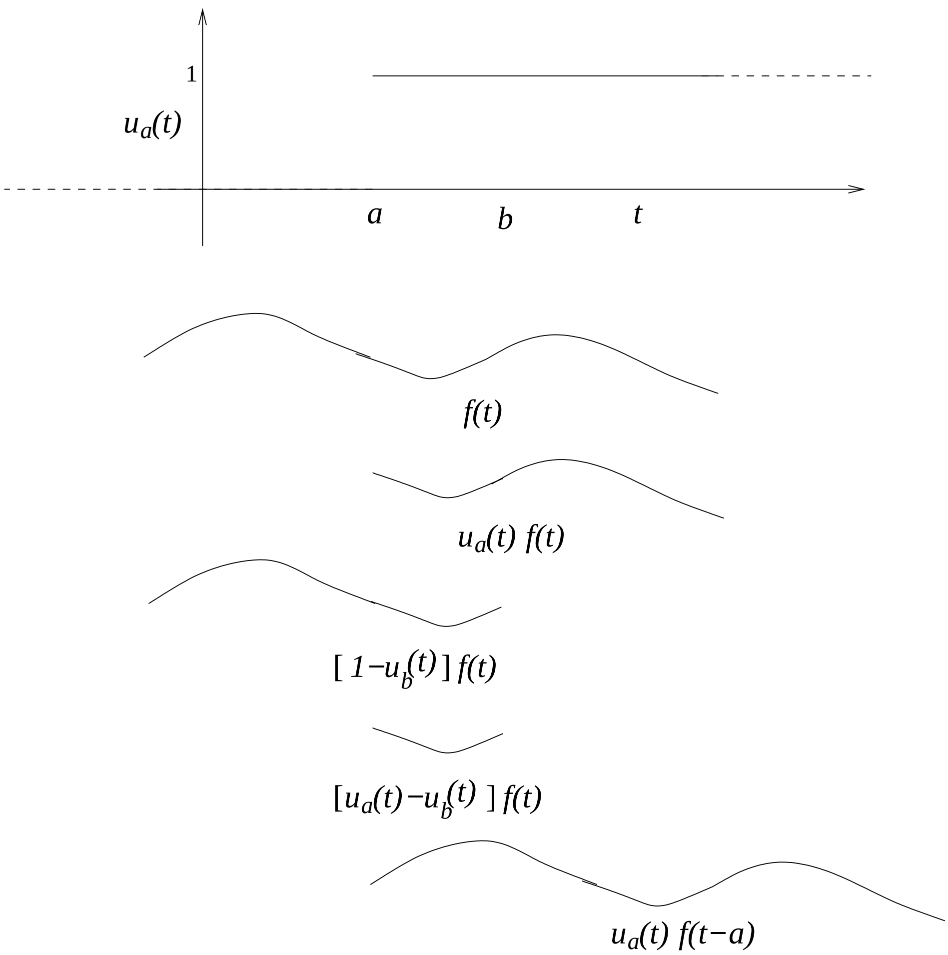


Figure 1: Effects of unit step function on a function $f(t)$. Here $b > a$.

**Example 1.** *Consider the function*

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ \sin 2t, & 1 < t \le \pi, \\ \cos t, & t > \pi \end{cases}$$

*Now let us consider a function g defined by*

$$g(t) = \Big(u_0(t) - u_1(t)\Big)t^2 + \Big(u_1(t) - u_\pi(t)\Big)\sin 2t + u_\pi(t)\cos t.$$

*Now $f(t)$ is piecewise continuous function. Hence, Laplace transform of $f$ exists. Clearly $f(t) = g(t)$ at all $t$ except possibly at a finite number points $t = 0, 1, \pi$ where $f(t)$ possibly has jump discontinuity. Hence, using Uniqueness Theorem of Laplace Transform (see Lecture Note 17), we conclude that $\mathcal{L}\Big(f(t)\Big) = \mathcal{L}\Big(g(t)\Big)$.*

**Theorem 1.** *(***Second shifting theorem***) If $\mathcal{L}\Big(f(t)\Big) = F(s)$, then*

$$\mathcal{L}\Big(u_a(t)f(t-a)\Big) = e^{-as}F(s).$$

*Conversely,*

$$\mathcal{L}^{-1}\Big(e^{-as}F(s)\Big) = u_a(t)f(t-a).$$

**Proof:** From the definition of Laplace transform

$$\begin{aligned}
\mathcal{L}\Big(u_a(t)f(t-a)\Big) &= \int_0^\infty e^{-st}u_a(t)f(t-a)\,dt \\
&= \int_a^\infty e^{-st}f(t-a)\,dt \\
&= e^{-as}\int_0^\infty e^{-su}f(u)\,du, \qquad t-a=u \\
&= e^{-as}F(s).
\end{aligned}$$

**Example 2.** *Find the Laplace transform of*

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ \sin 2t, & 1 < t \le \pi, \\ \cos t, & t > \pi \end{cases}$$

**Solution**: We know that if

$$g(t) = \Big(u_0(t) - u_1(t)\Big)t^2 + \Big(u_1(t) - u_\pi(t)\Big)\sin 2t + u_\pi(t)\cos t,$$

then $F(s) = G(s)$. Now we write $g(t)$ in such a way that second shifting theorem (see Theorem 1) can be applied. Hence, we manipulate $g(t)$ in the following way:

$$\begin{aligned}
g(t) &= u_0(t)t^2 - u_1(t)(t-1+1)^2 + u_1(t)\sin[2(t-1)+2] - u_\pi(t)\sin[2(t-\pi)] - u_\pi(t)\cos(t-\pi) \\
&= u_0(t)t^2 - u_1(t)(t-1)^2 - 2u_1(t)(t-1) - u_1(t) + \cos(2)u_1(t)\sin[2(t-1)] \\
&+ \sin(2)u_1(t)\cos[2(t-1)] + u_\pi(t)\sin[2(t-\pi)] - u_\pi(t)\cos(t-\pi)
\end{aligned}$$

Now every term is of the form $u_a(t)h(t-a)$. For example

$$u_0(t)t^2 \equiv u_0(t)(t-0)^2 \quad \text{and} \quad u_1(t) \equiv u_1(t)h(t-1) \quad \text{where } h(t) = 1.$$

Now we know that

$$\mathcal{L}\Big(1\Big) = \frac{1}{s},\ \mathcal{L}\Big(t\Big) = \frac{1}{s^2},\ \mathcal{L}\Big(t^2\Big) = \frac{2}{s^3},\ \mathcal{L}\Big(\cos t\Big) = \frac{s}{s^2+1},\ \mathcal{L}\Big(\sin t\Big) = \frac{1}{s^2+1}$$

and

$$\mathcal{L}\Big(\cos 2t\Big) = \frac{s}{s^2+4},\ \mathcal{L}\Big(\sin 2t\Big) = \frac{2}{s^2+4}$$

Hence,

$$F(s) = \frac{2}{s^3} - \frac{2e^{-s}}{s^3} - \frac{2e^{-s}}{s^2} - \frac{e^{-s}}{s} + \frac{2e^{-s}\cos 2}{s^2+4} + \frac{s\sin 2e^{-s}}{s^2+4} + \frac{2e^{-\pi s}}{s^2+4} - \frac{se^{-\pi s}}{s^2+1}$$

# 2 Laplace transform of derivatives and integrals

**Theorem 2.** *Let $f(t)$ be continuous for $t \geq 0$ and is of exponential order. Further suppose that $f$ is differentiable with $f'$ piecewise continuous in $[0, \infty)$. Then $\mathcal{L}\Big(f'\Big)$ exists and is given by*

$$\mathcal{L}\Big(f'\Big) = s\mathcal{L}\Big(f\Big) - f(0). \tag{1}$$

**Proof:** Since $f'$ is piecewise continuous in $[0, \infty)$, $f'$ is piecewise continuous in $[0, R]$ for any $R > 0$. Let $x_i$, $i = 0, 1, 2, \cdots, n$ are the possible points of jump discontinuity where $x_0 = 0$ and $x_n = R$. Now

$$\begin{aligned}
\int_0^R e^{-st} f'(t) &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} e^{-st} f'(t)\, dt \\
&= \sum_{i=0}^{n-1} e^{-st} f(t)\big|_{x_i}^{x_{i+1}} + s\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} e^{-st} f(t)\, dt \\
&= e^{-sR} f(R) - f(0) + s\int_0^R e^{-st} f(t)\, dt
\end{aligned}$$

Since $f$ is of exponential order, $|f(R)| \leq Me^{cR}$. This implies

$$|e^{-sR} f(R)| \leq Me^{-(s-c)R} \to 0 \quad \text{as} \quad R \to \infty \quad \text{for} \quad s > c.$$

Hence taking $R \to \infty$, we find

$$\mathcal{L}\Big(f'\Big) = s\mathcal{L}\Big(f\Big) - f(0). \tag{2}$$

**Corollary 1.** *Let $f$ and its derivatives $f^{(1)}, f^{(2)}, \cdots, f^{(n-1)}$ be continuous for $t \geq 0$ and are of exponential order. Further suppose that $f^{(n)}$ is piecewise continuous in $[0, \infty)$. Then Laplace transform of $f^{(n)}$ exists and is given by*

$$\mathcal{L}\Big(f^{(n)}\Big) = s^n\mathcal{L}\Big(f\Big) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{(n-1)}(0). \tag{3}$$

*In particular for $n = 2$, we get*

$$\mathcal{L}\Big(f''\Big) = s^2\mathcal{L}\Big(f\Big) - sf(0) - f'(0). \tag{4}$$

**Proof:** for $n = 2$, use (2) twice to find

$$\begin{aligned} \mathcal{L}\Big(f''\Big) &= s\mathcal{L}\Big(f'\Big) - f'(0) \\ &= s\Big(s\mathcal{L}\Big(f\Big) - f(0)\Big) - f'(0) \\ &= s^2\mathcal{L}\Big(f\Big) - sf(0) - f'(0). \end{aligned}$$

For general $n$, prove by induction.

**Example 3.** *Find Laplace transform of*

$$t\cos(\omega t).$$

**Solution**: Since $f(t) = t\cos(\omega t)$, we find

$$f'(t) = -\omega t\sin(\omega t) + \cos(\omega t)$$

and

$$f''(t) = -\omega^2 f(t) - 2\omega\sin(\omega t).$$

Hence taking Laplace transform on both sides, we find

$$\mathcal{L}\Big(f''\Big) = -\omega^2\mathcal{L}\Big(f\Big) - 2\omega\mathcal{L}\Big(\sin(\omega t)\Big)$$

Hence,

$$s^2\mathcal{L}\Big(f\Big) - sf(0) - f'(0) = -\omega^2\mathcal{L}\Big(f\Big) - 2\omega\frac{\omega}{s^2+\omega^2}.$$

Now $f(0) = 0, f'(0) = 1$. Simplifying, we find

$$\mathcal{L}\Big(f\Big) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

**Theorem 3.** *Let $F(s)$ be the Laplace transform of $f$. If $f$ is piecewise continuous in $[0,\infty)$ and is of exponential order, then*

$$\mathcal{L}\Big(\int_0^t f(\tau)\,d\tau\Big) = \frac{F(s)}{s}. \qquad (5)$$

**Proof:** Since $f$ is piecewise continuous,

$$g(t) = \int_0^t f(\tau)\,d\tau$$

is continuous. Since $f(t)$ is piecewise continuous, $|f(t)| \le Me^{kt}$ for all $t \ge 0$ except possibly at finite number of points where $f$ has jump discontinuities. Hence,

$$|g(t)| \le M\int_0^t e^{k\tau}\,d\tau = \frac{M}{k}(e^{kt}-1) \le \frac{M}{k}e^{kt}.$$

Thus, $g$ is continuous and is of exponential order. Hence, Laplace transform of $g$ exists. Further $g'(t) = f(t)$ and $g(0) = 0$. Using (2), we find

$$\mathcal{L}\Big(g'\Big) = s\mathcal{L}\Big(g\Big) - g(0) \implies \mathcal{L}\Big(g'\Big) = s\mathcal{L}\Big(g\Big) \implies G(s) = \frac{F(s)}{s}.$$

**Example 4.** *Find the inverse Laplace transform of* $1/s(s+1)^2$.

**Solution**: Since

$$\mathcal{L}\Big(t\Big) = \frac{1}{s^2} \implies \mathcal{L}\Big(te^{-t}\Big) = \frac{1}{(s+1)^2}$$

Hence for $f(t) = te^{-t}$, we have $F(s) = 1/(s+1)^2$. Thus,

$$\frac{1}{s(s+1)^2} = \frac{F(s)}{s} \implies \mathcal{L}^{-1}\left(\frac{1}{s(s+1)^2}\right) = \int_0^t \tau e^{-\tau}\,d\tau = 1-(t+1)e^{-t}$$

## 3 Derivative and integration of the Laplace transform

**Theorem 4.** *If* $F(s)$ *is the Laplace transform of* $f$*, then*

$$\mathcal{L}\Big(-tf(t)\Big) = F'(s), \quad \text{and} \quad \mathcal{L}^{-1}\Big(F'(s)\Big) = -tf(t). \tag{6}$$

**Comment:** The derivative formula for $F(s)$ can be derived by differentiating under the integral sign, i.e.

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}f(t)\right)\,dt \\ &= \int_0^\infty e^{-st}(-tf(t))\,dt \\ &= \mathcal{L}\Big(-tf(t)\Big). \end{aligned}$$

**Example 5.** *Consider the same problem as in Example 3, i.e. Laplace transform of* $t\cos(\omega t)$. *Let* $f(t) = \cos(\omega t)$. *Then*

$$F(s) = \frac{s}{s^2+\omega^2} \implies F'(s) = \frac{\omega^2 - s^2}{(s^2+\omega^2)^2}.$$

*Hence using (6), we find*

$$\mathcal{L}\Big(-t\cos(\omega t)\Big) = \frac{\omega^2-s^2}{(s^2+\omega^2)^2} \implies \mathcal{L}\Big(t\cos(\omega t)\Big) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

**Example 6.** *Find the inverse Laplace transform of*

$$F(s) = \ln\left(\frac{s-a}{s-b}\right)$$

**Solution**: If $\mathcal{L}\Big(f(t)\Big) = F(s)$, then $\mathcal{L}\Big(tf(t)\Big) = -F'(s)$. Hence

$$\mathcal{L}\Big(tf(t)\Big) = \frac{1}{s-b} - \frac{1}{s-a} = \mathcal{L}\Big(e^{bt}-e^{at}\Big) \implies f(t) = \frac{e^{bt}-e^{at}}{t}.$$

**Theorem 5.** *If $F(s)$ is the Laplace transform of $f$ and the limit of $f(t)/t$ exists as $t \to 0^+$, then*

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(p)\,dp, \qquad \textit{and} \qquad \mathcal{L}^{-1}\Big(\int_s^\infty F(p)\,dp\Big) = \frac{f(t)}{t}. \tag{7}$$

**Proof:** Let

$$g(t) = f(t)/t, \qquad \text{and} \qquad g(0) = \lim_{t\to 0^+}\frac{f(t)}{t}.$$

Now

$$F(s) = \mathcal{L}\Big(f(t)\Big) \implies F(s) = \mathcal{L}\Big(tg(t)\Big) = -G'(s), \qquad \text{[using (6)]}$$

Hence,

$$G(s) = \int_s^A F(p)\,dp.$$

Since $G(s) \to 0$ as $s \to \infty$, we must have

$$0 = \int_\infty^A F(p)\,dp$$

Thus,

$$G(s) = \int_s^A F(p)\,dp - \int_\infty^A F(p)\,dp \implies G(s) = \int_s^\infty F(p)\,dp \implies \mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(p)\,dp.$$

**Example 7.** *Find the Laplace transform of*

$$\frac{\sin\omega t}{t}.$$

**Solution**: Let $f(t) = \sin\omega t$. Using the formula (7), we find

$$\mathcal{L}\left(\frac{\sin\omega t}{t}\right) = \int_s^\infty \frac{\omega}{p^2+\omega^2}dp = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\omega}\right).$$

**Example 8.** *Consider the same problem as in Example 6, i.e. inverse Laplace transform of*

$$F(s) = \ln\left(\frac{s-a}{s-b}\right)$$

**Solution***: Note that*

$$\mathcal{L}\Big(f(t)\Big) = \ln\left(\frac{s-a}{s-b}\right) = \int_s^\infty \frac{1}{s-b}dp - \int_s^\infty \frac{1}{s-a}dp = \mathcal{L}\left(\frac{e^{bt}}{t}\right) - \mathcal{L}\left(\frac{e^{at}}{t}\right)$$

*Hence,*

$$\mathcal{L}\Big(f(t)\Big) = \mathcal{L}\left(\frac{e^{bt}-e^{at}}{t}\right) \implies f(t) = \frac{e^{bt}-e^{at}}{t}.$$