Lecture VII

Second order linear ODE, fundamental solutions, reduction of order

A second order linear ODE can be written as

$$y'' + p(x)y' + q(x)y = r(x), \qquad x \in \mathcal{I}, \tag{1}$$

where $\mathcal{I}$ is an interval. If $r(x) = 0,\ \forall\, x \in I$, then (1) is a homogeneous 2nd order linear ODE, otherwise it is non-homogeneous. We shall assume the following existence and uniqueness theorem for (1).

Theorem 1. *Let $p(x), q(x)$ and $r(x)$ be continuous in $\mathcal{I}$. If $x_0 \in \mathcal{I}$ and K_0, K_1 are two arbitrary real numbers, then (1) has unique solution $y(x)$ on $\mathcal{I}$ such that $y(x_0) = K_0$ and $y'(x_0) = K_1$.*

We shall also consider the homogeneous 2nd order linear equation

$$y'' + p(x)y' + q(x)y = 0, \qquad x \in \mathcal{I}. \tag{2}$$

Theorem 2. *Let $y_1(x)$ and $y_2(x)$ be two solutions of (2). Then $y(x) = c_1y_1(x)+c_2y_2(x)$ (c_1, c_2 arbitrary constants) is also a solution of (2).*

Proof: Trivial

Definition 1. *Two function f and g are defined in $\mathcal{I}$. If there exists constant a, b, not both zero such that*

$$af(x) + bg(x) = 0 \quad \forall x \in \mathcal{I},$$

then f and g are linearly dependent (LD) in $\mathcal{I}$, otherwise they are linearly independent (LI) in $\mathcal{I}$.

Example 1.
(i) $\sin x, \cos x,\ x \in (-\infty, \infty)$ *are LI.*
(ii) $x|x|, x^2,\ x \in (-1, 1)$ *are LI.*
(iii) $x|x|, x^2,\ x \in (0, 1)$ *are LD*

Definition 2. *Let f and g be two differentiable functions. Then*

$$W(f, g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

is called the Wronskian of f and g

Note : Let f and g be differentiable. If f and g are LD in an interval $\mathcal{I}$, then $W(f, g) = 0, \forall x \in \mathcal{I}$. Hence, if two differentiable functions f and g are such that $W(f, g) \neq 0$ at a point $x_0 \in \mathcal{I}$, then f and g are LI.
But the converse is not true. If $W(f, g) = 0, \forall x \in \mathcal{I}$, then f and g may not be LD. For example, consider $f(x) = x|x|, g(x) = x^2,\ x \in (-\infty, \infty)$. Here $W(f, g) = 0, \forall x$ but still f and g are LI.

Example 2. *For $f(x) = x, g(x) = \sin x$, we find $W(f, g) = x \cos x - \sin x$ which is nonzero, for example, at $x = \pi$. Hence, x and $\sin x$ are LI. Note that $W(f, g)$ may be zero at some point such as $x = 0$.*

Theorem 3. *Two solutions y_1, y_2 of (2) are LD <u>iff</u> $W(y_1, y_2) = 0$ at certain point $x_0 \in \mathcal{I}$.*

Proof: Let y_1, y_2 be LD. Thus, there exists a, b not both zero such that

$$ay_1(x) + by_2(x) = 0 \tag{3}$$

We can differentiate (3) once and obtain

$$ay_1'(x) + by_2'(x) = 0 \tag{4}$$

Now (3) and (4) can be viewed as linear homogeneous equations in two unknowns a and b. Since the solution is nontrivial, the determinant must be zero. Thus $W(y_1, y_2) = 0$, $\forall x \in \mathcal{I}$. Hence, $W(y_1, y_2)$ must be zero at $x_0 \in \mathcal{I}$.
Conversely, suppose $W(y_1, y_2) = 0$ at $x_0 \in \mathcal{I}$. Now consider

$$ay_1(x_0) + by_2(x_0) = 0 \tag{5}$$

and

$$ay_1'(x_0) + by_2'(x_0) = 0 \tag{6}$$

Now the determinant of the system of linear equations (in unknowns a, b) of (5) and (6) is the Wronskian $W(y_1, y_2)$ at x_0. Since, this is zero, we can find nontrivial solution for a and b. Take these nontrivial a and b and form

$$y(x) = ay_1(x) + by_2(x)$$

By (5) and (6), we find $y(x_0) = y'(x_0) = 0$. Hence, by uniqueness theorem $y(x) \equiv 0$, i.e. for nontrivial a and b

$$ay_1(x) + by_2(x) = 0, \qquad x \in \mathcal{I}$$

Hence y_1, y_2 are LD.
Comment: This theorem says that if f and g are solutions of (2) and $W(f, g) = 0$ at $x_0 \in \mathcal{I}$, then f and g must be LD. But in Example 2, $W(f, g) = 0$ at $x = 0$ but still f and g are LI. Do you find any contradiction in it?

Corollary 1. *Let y_1, y_2 be solutions of (2). If the Wronskian $W(y_1, y_2) = 0$ at $x_0 \in I$, then $W(y_1, y_2) = 0$ $\forall x \in \mathcal{I}$.*

Proof: We proceed as in the converse part of the previous theorem to prove that y_1 and y_2 are LD. Now proceed as in the first part of the same theorem to prove that $W(y_1, y_2) = 0$, $\forall x \in \mathcal{I}$.
Aliter: Since y_1 and y_2 are solutions of (2), we obtain

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \tag{7}$$
$$y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{8}$$

Multiply (7) by y_2 and (8) by y_1 and subtract. This leads to

$$\frac{dW}{dx} + p(x)W = 0,$$

where we have used the short notation W for $W(y_1, y_2)$. Integrating, we find

$$W(x) = Ce^{-\int p(x)\,dx}$$

Since $W(x_0) = 0$, this gives $C = 0$ and hence $W \equiv 0$.

Theorem 4. *Let y_1, y_2 be solutions of (2). If there exists a point $x_0 \in \mathcal{I}$ such that $W(y_1, y_2) \neq 0$ at x_0, then y_1 and y_2 are LI and forms a basis solution for (2).*

Proof: If y_1 are y_2 are LD, then $W(y_1, y_2) \equiv 0$ which contradicts $W(y_1, y_2) \neq 0$ at x_0. Hence, y_1 and y_2 are LI.

Now we shall show that y_1 and y_2 spans the solution space for (2). Let y be any solution with $y(x_0) = K_0$ and $y'(x_0) = K_1$. Now, the system

$$\begin{aligned} ay_1(x_0) + by_2(x_0) &= K_0 \\ ay_1'(x_0) + by_2'(x_0) &= K_1 \end{aligned}$$

has unique solution $a = c_1$ and $b = c_2$, since the determinant is nonzero. Let $\zeta(x) = c_1y_1(x) + c_2y_2(x)$. Then, $\zeta(x_0) = K_0$, $\zeta'(x_0) = K_1$. But by the existence and uniqueness theorem, we have $y(x) \equiv \zeta(x)$ and thus

$$y(x) = c_1y_1(x) + c_2y_2(x), \quad \forall x \in \mathcal{I}$$

Hence, y_1 and y_2 spans the solution space. Thus, y_1 and y_2 form a basis of solution for (2). Thus, a *general solution* $y(x)$ of (2) can be written as

$$y(x) = Ay_1(x) + By_2(x),$$

where A and B are arbitrary constants. For an IVP, these constants take particular values to satisfy the initial condition.

Existence of basis: By the existence and uniqueness theorem, there exists a solution $y_1(x)$ of (2) with $y_1(x_0) = 1, y_1'(x_0) = 0$. Similarly, there exists a solution $y_2(x)$ of (2) with $y_2(x_0) = 0, y_2'(x_0) = 1$. Hence, $W(y_1, y_2) = 1 \neq 0$ at x_0. By the previous, theorem y_1 and y_2 form a basis solution for (2).

Example 3. $y_1(x) = \sin x$ *and* $y_2(x) = \cos x$ *satisfy* $y''+y = 0$ *and* $W(y_1, y_2) = -1 \neq 0$. *Hence,* $\sin x$ *and* $\cos x$ *form a basis of solution for* $y'' + y = 0$. *Thus, a general solution of* $y'' + y = 0$ *is* $y(x) = C_1 \sin x + C_2 \cos x$.

Reduction of order: Consider the homogeneous 2nd order linear equation

$$y'' + p(x)y' + q(x)y = 0. \tag{9}$$

If we know one nonzero solution $y_1(x)$ (by any method) of (9), then it is easy to find the second solution $y_2(x)$ which is independent of y_1. Thus, y_1 and y_2 will form a basis of solution.

We assume that $y_2(x) = v(x)y_1(x)$, where $v(x)$ is an unknown function. Since, y_2 is a solution, we substitute $y_2(x) = v(x)y_1(x)$ into (9). Taking into account the fact that y_1 is also a solution of (9), we find

$$y_1v'' + (2y_1' + py_1)\,v' = 0.$$

Dividing this by y_1 and writing U for v', we get

$$U' + \left(\frac{2y_1'}{y_1} + p\right) U = 0$$

Since this is linear equation, it has general solution

$$U = \frac{C}{y_1^2} e^{-\int p\, dx},$$

where C is a constant of integration. Thus, we find

$$v(x) = C \int \frac{1}{y_1^2} e^{-\int p\, dx} + D,$$

where D is another constant of integration. Finally, multiply v by y_1 to find y_2:

$$y_2(x) = C y_1(x) \int \frac{1}{y_1^2} e^{-\int p\, dx} + D y_1(x).$$

Since, we are looking for a solution independent of y_1, this can be taken with $C = 1$ and $D = 0$. Thus

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2} e^{-\int p\, dx}.$$

To show that they are LI, note that

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = y_1^2 v' = y_1^2 U = e^{-\int p\, dx} \neq 0.$$

Thus, y_1 and y_2 form a basis of solution.

Example 4. *Solve* $xy'' + (2x+1)y' + (x+1)y = 0$

Solution: Since at $x = 0$, the equation becomes singular, we solve the above for $x \neq 0$. WLOG, we assume that $x > 0$. Clearly, $y_1(x) = e^{-x}$ is a solution. We write this equation as

$$y'' + \left(2 + \frac{1}{x}\right) y' + \frac{x+1}{x} y = 0.$$

Hence, $p(x) = 2 + 1/x$. Substituting $y_2(x) = v(x) y_1(x)$ and solving we find

$$v(x) = \int \frac{1}{e^{-2x}} \exp\left(-\int (2 + 1/x)\, dx\right) = \ln x$$

Hence, $y_2(x) = e^{-x} \ln x$. Thus, the general solution is $y(x) = e^{-x}(C_1 + C_2 \ln x), \quad x > 0$. What is the general solution for $x < 0$?