Lecture IX

Non-homogeneous linear ODE, method of undetermined coefficients

1 Non-homogeneous linear equation

We shall mainly consider 2nd order equations. Extension to the n-th order is straight forward.
Consider a 2nd order linear ODE of the form

$$y'' + p(x)y' + q(x)y = r(x), \qquad x \in \mathcal{I}, \tag{1}$$

where p, q are continuous functions. Let $y_p(x)$ is a (particular) solution to (1). Then

$$y_p'' + p(x)y_p' + q(x)y_p = r(x).$$

Let y be any solution to (1). Now consider $Y = y - y_p$ satisfies

$$Y'' + p(x)Y' + q(x)Y = 0.$$

Thus, Y satisfies a homogeneous linear 2nd order ODE. Hence, we express Y as linear combination of two LI solutions y_1 and y_2. This gives

$$Y = C_1y_1 + C_2y_2$$

or

$$y = C_1y_1 + C_2y_2 + y_p \tag{2}$$

Thus, the general solution to (1) is given by (2). We have seen how to find y_1 and y_2. Here, we concentrate on methods to find y_p.

1.1 Method of undetermined coefficients

This method works for the following nonhomogeneous linear equation:

$$ay'' + by' + cy = r(x), \qquad x \in \mathcal{I}, \tag{3}$$

where a, b, c are constants and $r(x)$ is a finite linear combination of products formed from the polynomial, exponential and sines or cosines functions. Thus, $r(x)$ is a finite linear combination of functions of the following form:

$$e^{\alpha x}x^m \left\{ \begin{array}{l} \sin\beta x \\ \cos\beta x \end{array} \right. ,$$

where m is a nonnegative integer.
Suppose

$$r(x) = r_1(x) + r_2(x) + \cdots + r_n(x).$$

If $y_{pi}(x),\ 1 \le i \le n$ is a particular solution to

$$ay'' + by' + cy = r_i(x),$$

then it is trivial to prove that

$$y_p(x) = y_{p1}(x) + y_{p2}(x) + \cdots + y_{pn}(x)$$

is a particular solution to

$$ay'' + by' + cy = r(x).$$

Hence, we shall consider the case when $r(x)$ is one of the $r_i(x)$. Thus, we choose $r(x)$ to be of the following form:

$$e^{\alpha x} x^m \begin{cases} \sin \beta x \\ \cos \beta x \end{cases}.$$

Rule 1: If none of the terms in $r(x)$ is a solution of the homogeneous problem, then for y_p, choose a linear combination of $r(x)$ and all its derivatives that form a finite set of linearly independent functions.

Example 1. *Consider*

$$y'' - 2y' + 2y = x \sin x.$$

Solution: The LI solutions of the homogeneous part are $e^x \cos x$ and $e^x \sin x$. Clearly, neither x nor $\sin x$ is a solution of the homogeneous part. Hence, we choose

$$y_p(x) = ax \sin(x) + bx \cos x + c \cos x + d \sin x.$$

Now substituting into the governing equation, we get

$$(a+2b)x \sin x + (-2a+b)x \cos x + (2a-2b-2c+d) \cos x + (-2a-2b+c+2d) \sin x = x \sin x.$$

Hence

$$a + 2b = 1, \quad -2a + b = 0, \quad (2a - 2b - 2c + d) = 0, \quad (-2a - 2b + c + 2d) = 0.$$

Solving, we get

$$a = \frac{1}{5},\ b = \frac{2}{5},\ c = \frac{2}{25},\ d = \frac{14}{25}$$

Hence, the general solution is

$$y = e^x(C_1 \cos x + C_2 \sin x) + \frac{1}{5} x \sin(x) + \frac{2}{5} x \cos x + \frac{2}{25} \cos x + \frac{24}{25} \sin x$$

Aliter: (*Annihilator method*) Writing $D \equiv d/dx$, we write

$$(D^2 - 2D + 2)y_p = x \sin x.$$

Note that $(D^2 + 1)^2 x \sin x = 0$. Hence, operating $(D^2 + 1)^2$ on both sides, we find

$$(D^2 + 1)^2 (D^2 - 2D + 2)y_p = 0.$$

The characteristic roots are found from $(m^2 + 1)^2(m^2 - 2m + 2) = 0$. Thus, $m = -1 \pm i$ and $m = \pm i, \pm i$. Now solution to this homogeneous linear ODE with constant coefficient is

$$y_p = e^x(c_1 \cos x + c_2 \sin x) + (c_3 \cos x + c_4 \sin x) + x(c_5 \cos x + c_6 \sin x)$$

Since, the first two terms are the solution of the original homogeneous part and hence contribute nothing. Thus, the form for y_p must be

$$y_p = (c_3 \cos x + c_4 \sin x) + x(c_5 \cos x + c_6 \sin x),$$

which conforms with previous form.

Rule 2: If $r(x)$ contains terms that are solution of the homogeneous linear part, then to choose the trial form of y_p follow the following steps. First, choose a linear combination of $r(x)$ and its derivatives which are LI. Second, this linear combination is multiplied by a power of x, say x^k, where k is the smallest nonnegative integer that makes all the new terms not to be solutions of the homogeneous problem.

Example 2. *Consider*

$$y'' - 2y' - 3y = xe^{-x}.$$

Solution: The LI solutions of the homogeneous part are e^{-x} and e^{3x}. Clearly, e^{-x} is a solution of the homogeneous part. Hence, we choose $y_p(x) = x(axe^{-x} + be^{-x})$. Substituting, we find

$$e^{-x}(-4b + 2a - 8ax) = xe^{-x}$$

This, gives $-4b + 2a = 0, -8a = 1$ and thus $a = -1/8, b = -1/16$. Thus, the general solution is

$$y = C_1 e^{-x} + C_2 e^{3x} - \frac{xe^{-x}}{16}(2x + 1)$$

Aliter: (*Annihilator method*) Writing $D \equiv d/dx$, we write

$$(D^2 - 2D - 3)y_p = xe^{-x}.$$

Since $(D + 1)^2 xe^{-x} = 0$, operating $(D + 1)^2$ on both sides we find

$$(D + 1)^2(D^2 - 2D - 3)y_p = 0$$

The characteristic roots are found from $(m + 1)^2(m^2 - 2m - 3) = 0$. Thus, $m = -1, -1, -1, 3$. Now solution to this homogeneous linear ODE with constant coefficient is

$$y_p = c_1 e^{3x} + e^{-x}(c_2 + c_3 x + c_4 x^2)$$

Since, the first two terms are the solution of the original homogeneous part and hence contribute nothing. Thus, the form for y_p must be

$$y_p = e^{-x}(c_3 x + c_4 x^2),$$

which conforms with the previous form.

Example 3. *Consider*

$$y'' - 2y' + y = 6xe^x$$

Solution: The LI solutions of the homogeneous part are e^x and xe^x. Clearly, both e^x, xe^x are solutions of the homogeneous part. Hence, we choose $y_p(x) = x^2(axe^x + be^x)$. Substituting, we find

$$e^x(2b + 3ax) = 6xe^x$$

This, gives $a = 1, b = 0$. Thus, the general solution is

$$y = e^x(C_1 + C_2 x + x^3)$$

Example 4. *Consider*

$$y''' - 3y'' + 2y' = 10 + 4xe^{2x}.$$

Solution: The LI solutions of the homogeneous part related to the characteristic equation

$$m^3 - 3m^2 + 2m = 0 \quad \Rightarrow \quad m = 0, 1, 2.$$

Thus the LI solutions are $1, e^x$ and e^{2x}. Clearly, e^{2x} and 10 are solutions of the homogeneous part. Hence, we choose $y_p(x) = ax + x(bxe^{2x} + ce^{2x})$. Substituting, we find

$$2a + (6b + 2c)e^{2x} + 4bxe^{2x} = 10 + 4xe^{2x}$$

This, gives $a = 5, b = 1$ and $c = -3$ Thus, the general solution is

$$y = C_1 + C_2e^x + C_3e^{2x} + 5x + e^{2x}(x^2 - 3x)$$

Aliter: (*Annihilator method*) Writing $D \equiv d/dx$, we write

$$(D^3 - 3D + 2D)y_p = 10 + 4xe^{2x}.$$

To annihilate 10 we apply D and to annihilate xe^{2x}, we apply $(D - 2)^2$. Thus,

$$D(D - 2)^2(D^3 - 3D + 2D)y_p = 0$$

The characteristic roots are found from $m(m - 2)^2(m^3 - 3m + 2m) = 0$. Thus, $m = 0, 0, 1, 2, 2, 2$. Now solution to this homogeneous linear ODE with constant coefficient is

$$y_p = c_1 + c_2x + c_3e^x + e^{2x}(c_4 + c_5x + c_6x^2)$$

The terms with c_1, c_3 and c_4 are the solution of the original homogeneous part and hence contribute nothing. Thus, the form for y_p must be

$$y_p = c_2x + e^{2x}(c_5x + c_6x^2).$$

which conforms with the previous form.