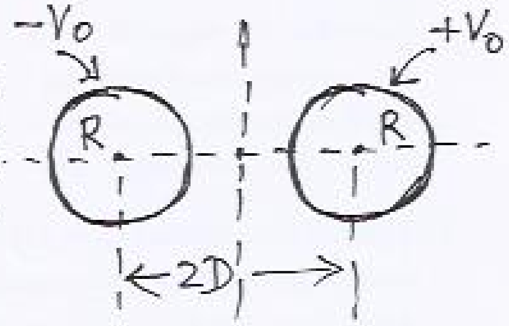
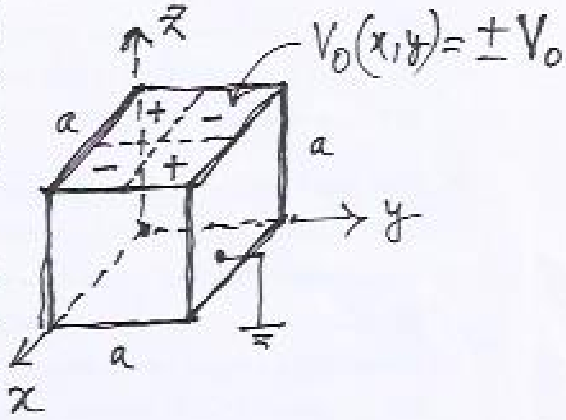
IITK / PHY 552A / Mid-Semester Exam / 22 Sep 2018 / AS /
Marks: 80 / Time: 2 hrs.



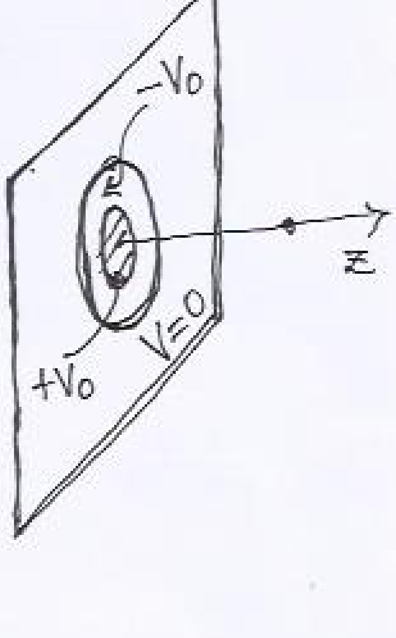
1. Two long, straight copper cylinders, each of radius R, and held a distance $2D$ apart, are at potentials $+V_0$ and $-V_0$ as shown. The potential in the surrounding region can be determined from an equivalent image problem involving two line charges $+\lambda$ and $-\lambda$. Determine λ in terms of V_0, D, R. [10]



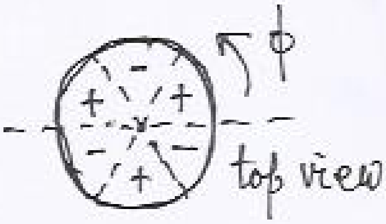
2. A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded. The potential $V_0(x,y)$ on the top surface ($z=a$), which is insulated from the plates, is as shown. Find the potential inside the box. [Suggestion: Factorize the boundary condition $V_0(x,y) = V_0\tau(x)\tau(y)$ using appropriate sign functions.] [12]

3. The potentials on the surfaces of two concentric spheres of radii R and $2R$ are given by: $V_{\text{inner}}(\theta) = V_0\cos\theta$ and $V_{\text{outer}}(\theta) = V_0(3\cos^2\theta - 1)/2$, respectively. Obtain the potential $V(r,\theta)$ for: (i) $r < R$, (ii) $r > 2R$, and (iii) $R < r < 2R$. [12]

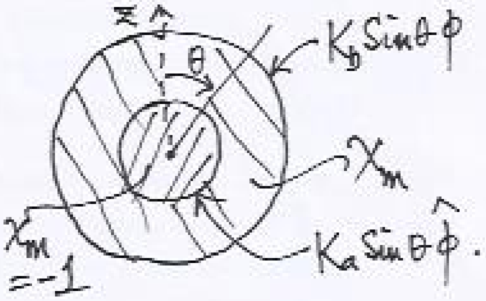
4. A thick dielectric spherical shell (inner and outer radii a and b, electric susceptibility χ_e, $\rho_{\text{free}} = 0$) is placed in an otherwise uniform electric field $E_0\hat{z}$. (a) What are the field-induced charge densities (bulk and surface) ? (b) Find the potential $V(r,\theta)$ inside the dielectric shell. (c) Determine σ_a in terms of the given quantities and write the expression in a simplified form. [12]



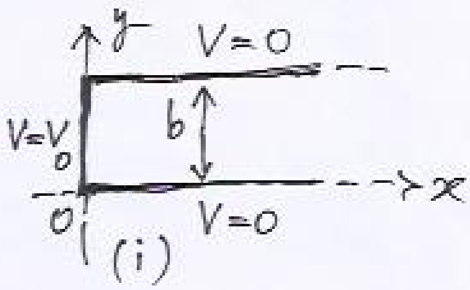
5. The boundary condition specified on the $x-y$ plane is as shown. The potential $V = +V_0$ over a circular region of radius R and $V = -V_0$ from radius R to $\sqrt{2}R$. Everywhere else $V = 0$. There are no charges in the region of interest $z > 0$. (a) Using the Green's function method find the potential $V(z)$ at a point on the positive z-axis. (b) Expand $V(z)$ in powers of R/z up to 4^{th} order. (c) Extend the result and obtain $V(r,\theta,\phi)$ for off-axis points. [10]

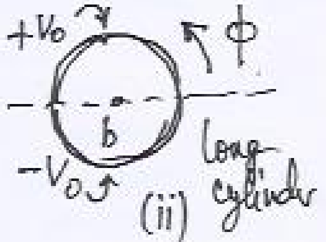


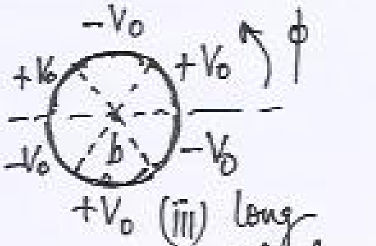
6. The potential $V_0(\theta, \phi)$ on the surface of a sphere of radius R is alternately $+V_0$ and $-V_0$ on the six segments in the upper hemisphere (as shown) and $-V_0$ and $+V_0$ (opposite sign) on the corresponding six segments in the lower hemisphere. Obtain the first non-zero contribution (in the series $\sum_{l,m}$) to the potential $V(r, \theta, \phi)$ outside the sphere. Overall constant is not required. Provide brief justification. [6]



7. A perfect diamagnetic ($\chi_m = -1$) sphere of radius a is covered by a spherical shell (magnetic susceptibility χ_m) up to radius b, and placed in an otherwise uniform magnetic field $B_0\hat{z}$. (a) Determine $K_a + K_b$. (b) Including all contributions, obtain the component B_θ of the magnetic field inside the shell at (r, θ). (c) Obtain K_b in terms of B_θ at $r = b$. [10] $[\mathbf{K}_{\text{mag}} = K_{a/b} \sin\theta\ \hat{\phi}]$







8. The two boundary-value problems (i) and (ii) involving the two-dimensional Laplace equation can be mapped to each other by appropriate change of variables. (a) Write the expressions for the solutions $V(x, y)$ and $V(\rho, \phi)$ corresponding to (i) and (ii). (b) Considering simple extensions of (i), and the mapping, obtain the potential $V(\rho, \phi)$ outside a long cylinder with the boundary condition shown in (iii). [8]

Useful information:

$$\frac{2(\rho^2 - b^2)/b}{\rho^2 + b^2 - 2\rho b \cos(\phi' - \phi)}$$

$$\frac{2z}{[(x - x')^2 + (y - y')^2 + z^2]^{3/2}}$$

$$\int_0^{2\pi} \frac{\cos m\phi'' d\phi''}{1 + \beta^2 - 2\beta \cos\phi''} = \frac{2\pi\beta^m}{1 - \beta^2} \quad (\beta < 1)$$

$$\sum_{n \text{ odd}} \frac{1}{n} \sin n\phi\, e^{-n\xi} = \frac{1}{2} \tan^{-1}\left(\frac{\sin\phi}{\sinh\xi}\right)$$

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

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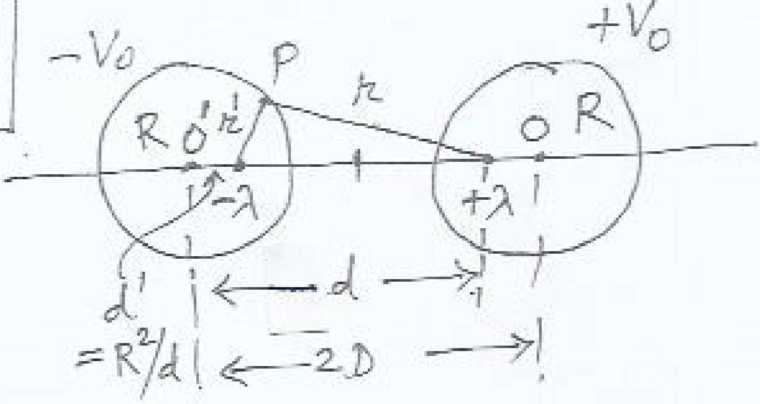
Brief solutions.

Q1.



$$-V_0 = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r'}{r}$$

$$\left.\frac{r'}{r}\right|_{P \text{ on circle}} = R/d.$$

$$\Rightarrow \boxed{V_0 = \frac{\lambda}{2\pi\epsilon_0} \cdot \ln \frac{d}{R}} \quad \text{—— (1)}$$

$$O'O = 2D = d + d' = d + R^2/d$$

$$\Rightarrow \frac{2D}{R} = \frac{d}{R} + \frac{R}{d} \equiv e^{X} + e^{-X} \qquad \left(\frac{d}{R} \equiv e^{X}\right).$$

$$\Rightarrow \cosh X = \frac{D}{R} \Rightarrow \boxed{X = \cosh^{-1}\frac{D}{R}} \quad \text{—— (2)}$$

determines X hence d.

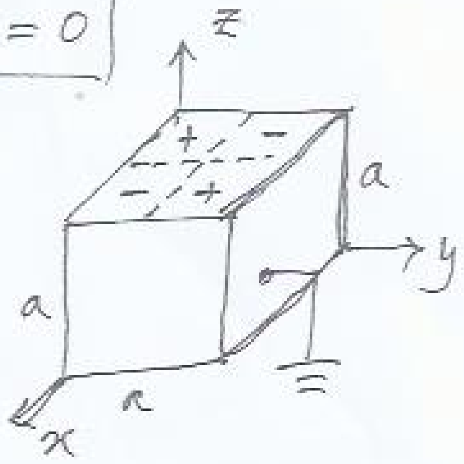
$$\text{Thus, } V_0 = \frac{\lambda}{2\pi\epsilon_0} \cdot \ln . e^{X} = \frac{\lambda}{2\pi\epsilon_0} \cdot X$$

$$\Rightarrow \boxed{\lambda = \frac{2\pi\epsilon_0 . V_0}{\cosh^{-1} D/R}} \quad \text{—— (3)}.$$

[10].

Q2.

$$V(x,y,z) = X(x)\cdot Y(y)\cdot Z(z) \qquad \boxed{\nabla^2 V = 0}$$



$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

$$\underset{-k_x^2}{\parallel} \qquad \underset{-k_y^2}{\parallel} \qquad (k_x^2 + k_y^2).$$

$$\left.\begin{array}{l} X(x) = A_m \sin\frac{m\pi x}{a} \\ Y(y) = B_n \sin\frac{n\pi y}{a} \end{array}\right\} \boxed{V=0 \text{ at } x=0,a \text{ and } y=0,a}$$

$$Z(z) = F_{m,n} e^{Kz} + G_{m,n} e^{-Kz} = 0 \text{ for } z=0 \Rightarrow G=-F.$$

$$\sim \sinh Kz. \qquad \left(K = \sqrt{(m^2+n^2)}\,\pi/a\right)$$

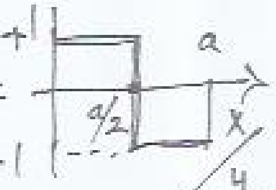
$$\Rightarrow \boxed{V(x,y,z) = \sum_{m,n} A_{m,n} \sin\frac{m\pi x}{a}\cdot \sin\frac{n\pi y}{a}\cdot \sinh\sqrt{m^2+n^2}\,\frac{\pi}{a} z} \qquad [6]$$

Apply bd. cond^n at $z=a$:

$$\int_0^a\int_0^a V_0(x,y)\left(\sin\frac{m\pi}{a}x\right)\left(\sin\frac{n\pi}{a}y\right)dx\,dy = \left(\frac{a}{2}\right)^2\cdot A_{m,n}\cdot \sinh\sqrt{m^2+n^2}\,\pi$$

$$\text{LHS} = V_0\int_0^a dx\cdot \tau(x)\left(\sin\frac{m\pi}{a}x\right)\cdot\int_0^a dy\,\tau(y)\left(\sin\frac{n\pi y}{a}\right)$$

$$= V_0\left[\int_0^{a/2} dx\left(\sin\frac{m\pi}{a}x\right) - \int_{a/2}^a dx\left(\sin\frac{m\pi}{a}x\right)\right]\times\left[x\to y\right] \quad \text{where } \begin{array}{l}\tau(x)\\ \tau(y)\end{array} =$$



$$= V_0\cdot\left(\frac{4a}{m\pi}\right)\left(\frac{4a}{n\pi}\right) \qquad \boxed{\begin{array}{l}\text{where } m = 2, 6, 10 \ldots \\ \phantom{\text{where }} n = 2, 6, 10 \ldots\end{array}}$$

$4K-2,\ K=1,2,3$

$4n+2,\ n=0,1,2$

$$\Rightarrow A_{m,n} = V_0\cdot\left(\frac{4a}{m\pi}\right)\left(\frac{4a}{n\pi}\right)\cdot\left(\frac{2}{a}\right)^2 \Big/ \sinh\sqrt{m^2+n^2}\cdot\pi$$

$$\boxed{V_0\cdot\frac{64}{\pi^2}\cdot\frac{1}{m}\cdot\frac{1}{n}\Big/ \quad ''} \qquad [6]$$

Q3.

(i) $r<R$: $V(r,\theta) = \sum_{l=0,1,2\cdots} A_l r^l P_l(\cos\theta)$ $\quad (B_l = 0)$.

As $r \to R$, $V(R,\theta) = V_0 \cos\theta = V_0 P_1(\cos\theta)$

$\Rightarrow A_1 = V_0/R$, all other coefficients $= 0$.

So $\boxed{V(r,\theta) = V_0 \cdot \frac{r}{R} \cos\theta}$ ——— [2]

(ii) $r>2R$: $V(r,\theta) = \sum_l F_l/r^{l+1} \cdot P_l(\cos\theta)$ $\quad (E_l = 0)$.

As $r \to 2R$, $V(2R,\theta) = V_0 P_2(\cos\theta) \Rightarrow F_2 = V_0(2R)^3$, all other coeff. $= 0$.

So $\boxed{V(r,\theta) = V_0 \left(\frac{2R}{r}\right)^3 \cdot P_2(\cos\theta).}$ ——— [2]

(iii) $R<r<2R$: $V(r,\theta) = \sum_l \left(C_l r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos\theta)$.

Only $l = 1, 2$ terms will survive $\longrightarrow$ $\left\{\begin{array}{l} \text{For } l \neq 1,2 \\ r \to R: \ 0 = C_l R^l + \frac{D_l}{R^{l+1}} \\ r \to 2R: \ 0 = C_l (2R)^l + \frac{D_l}{(2R)^{l+1}} \end{array}\right\} \Rightarrow C_l, D_l = 0$

For $l = 1, 2$:

As $r \to R$: $\begin{cases} V_0 = C_1 R + \frac{D_1}{R^2} & (l=1) \\ 0 = C_2 R^2 + \frac{D_2}{R^3} & (l=2) \end{cases}$

Similarly, $r \to 2R$: $\begin{cases} 0 = C_1 (2R) + \frac{D_1}{(2R)^2} & (l=1) \\ V_0 = C_2 (2R)^2 + \frac{D_2}{(2R)^3} & (l=2) \end{cases}$

Solving for C_1, D_1 & C_2, D_2

$\Rightarrow$

$$V(r,\theta) = V_0 \left[\left\{ -\frac{1}{7}\frac{r}{R} + \frac{8}{7}\left(\frac{R}{r}\right)^2 \right\} P_1(\cos\theta) + \left\{ \frac{8}{31}\left(\frac{r}{R}\right)^2 - \frac{8}{31}\left(\frac{R}{r}\right)^3 \right\} P_2(\cos\theta) \right].$$

——— [8].

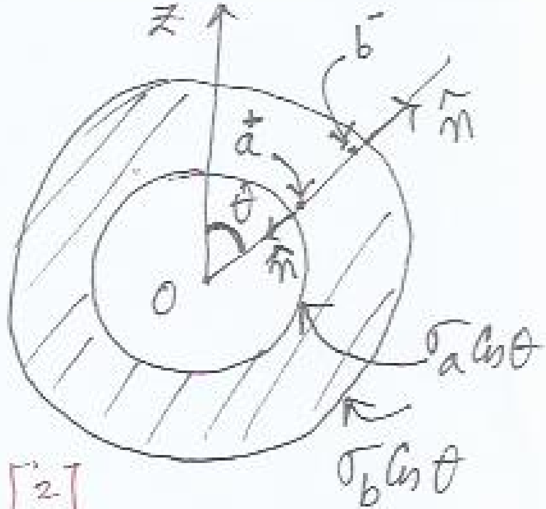
Q4

As $\rho_{free} = 0$

$\vec{\nabla}\cdot\vec{D} = 0 \Rightarrow \vec{\nabla}\cdot\vec{P} = 0$ $\left\{ \begin{array}{l} \vec{P} = \epsilon_0 \chi_e \vec{E} \\ \vec{D} = \epsilon_0 \vec{E} + \vec{P} \end{array} \right\}$

$\Rightarrow \rho_{pol} = 0$.

Only surface charge densities $\left\{ \begin{array}{ll} \sigma_a \cos\theta & \text{(inner)} \\ \sigma_b \cos\theta & \text{(outer)}. \end{array} \right.$



[2]

$V(r,\theta)$ in the dielectric shell $= \left[-E_0 r\cos\theta + \frac{\sigma_a}{3\epsilon_0}\frac{a^3}{r^2}\cos\theta + \frac{\sigma_b}{3\epsilon_0} r\cos\theta \right]$

[3]

Find P_r at $r \to b^-$ and $r \to a^+$ and connect to σ_b and σ_a $\left\{ \begin{array}{l} \text{just inside outer} \\ \text{and } \text{~~just outside~~ inner} \\ \text{surfaces of the dielectric} \end{array} \right.$

$r \to b^-$, $P_r = \epsilon_0 \chi_e E_r \Big|_b = \vec{P}\cdot\hat{n} = \sigma_{pol}\Big|_b = \sigma_b \cos\theta$.

$$\Rightarrow \sigma_b \cos\theta = \epsilon_0 \chi_e \left[E_0 + \frac{\sigma_a}{3\epsilon_0}\cdot\frac{2a^3}{b^3} - \frac{\sigma_b}{3\epsilon_0} \right] \cos\theta$$

$$\Rightarrow \boxed{\sigma_b \left(1 + \frac{\chi_e}{3}\right) = \epsilon_0 \chi_e \left[E_0 + \frac{\sigma_a}{3\epsilon_0}\frac{2a^3}{b^3} \right]} \quad \text{—— (1)}$$

Similarly, for $r \to a^+$, $P_r = \epsilon_0 \chi_e E_r \Big|_a = \vec{P}\cdot(-\hat{n}) = -\sigma_a \cos\theta$.

$$\Rightarrow -\sigma_a \cos\theta = \epsilon_0 \chi_e \left[E_0 + \frac{\sigma_a}{3\epsilon_0}\cdot 2 - \frac{\sigma_b}{3\epsilon_0} \right] \cos\theta$$

[7]

$$\Rightarrow \boxed{\sigma_b \frac{\chi_e}{3} = \sigma_a \left(1 + \frac{2}{3}\chi_e\right) + \epsilon_0 \chi_e E_0} \quad \text{—— (2)}$$

$$(1) - (2) \Rightarrow \sigma_b = -\sigma_a \left[1 + \frac{2\chi_e}{3}\left(1 - a^3/b^3\right) \right]$$

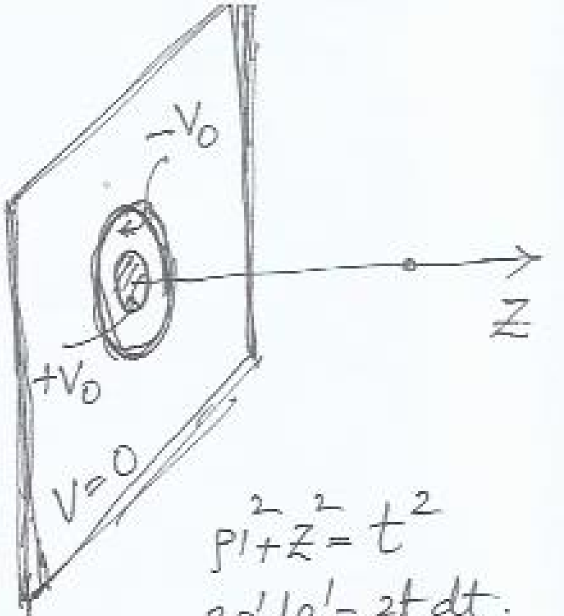
and from (2):

$$\sigma_a = \frac{-\epsilon_0 \chi_e E_0}{\left[(1+\chi_e) + \frac{\chi_e}{3}\cdot\frac{2\chi_e}{3}\left(1 - \frac{a^3}{b^3}\right) \right]}$$

Q5

$$V(z) = -\frac{1}{4\pi}\oint_S V_0(x',y')\,\frac{\partial G}{\partial n'}\cdot dx'dy'$$

(a)



$$= \frac{1}{4\pi}\int V_0(\rho')\cdot\frac{2z}{(\rho'^2+z^2)^{3/2}}\,2\pi\rho'd\rho' \quad (x,y=0).$$

$$= V_0\cdot z\left[\int_0^R - \int_R^{\sqrt{2}R}\left\{\frac{\rho'd\rho'}{(\rho'^2+z^2)^{3/2}}\right\}\right]$$

$\rho'^2+z^2=t^2$

$2\rho'd\rho' = 2t\,dt.$

$$\int\frac{\rho'd\rho'}{(\rho'^2+z^2)^{3/2}} = \int\frac{t\,dt}{t^3} = -\frac{1}{t} = -\frac{1}{\sqrt{\rho'^2+z^2}}$$

Hence, $V(z) = V_0\cdot z\left[\frac{1}{\sqrt{\rho'^2+z^2}}\Big|_R^0 - \frac{1}{\sqrt{\rho'^2+z^2}}\Big|_{\sqrt{2}R}^R\right]$

$$= V_0 z\left[\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)-\left(\frac{1}{\sqrt{R^2+z^2}}-\frac{1}{\sqrt{2R^2+z^2}}\right)\right] \quad [6]$$

$$= V_0\left[1-\frac{2}{(1+R^2/z^2)^{1/2}}+\frac{1}{(1+2R^2/z^2)^{1/2}}\right]$$

(b) Expansion upto 4th order in powers of (R/z):

$$V(z) \simeq V_0\left[1-2\left\{1-\frac{1}{2}\frac{R^2}{z^2}+\frac{3}{8}\frac{R^4}{z^4}\right\}+\left\{1-\frac{R^2}{z^2}+\frac{3}{2}\frac{R^4}{z^4}\right\}\right]\cdots$$

$$= V_0\left[\left(\frac{3}{2}-\frac{3}{4}\right)\frac{R^4}{z^4}\right] = \frac{3V_0}{4}\left(R/z\right)^4 \quad [2]$$

(c) Hence, extending to points off-axis, $V(r,\theta) \simeq \frac{3V_0}{4}\cdot\left(\frac{R}{r}\right)^4\cdot P_3(\cos\theta) \quad [2]$

Q6 - * Since $V_0(\theta,\phi)$ changes sign every 60° with increasing ϕ, the dominant ϕ-dependent term will be $\sin 3\phi$ $(m=3)$. Hence $l \geq 3$.

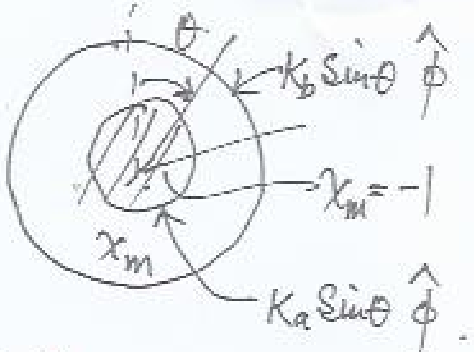
* For $l=3$, $m=3$, $P_l^m(\cos\theta) \sim \sin^3\theta$, which is even about $\theta = \pi/2$, whereas $V_0(\theta,\phi)$ changes sign at $\theta = \pi/2$. $\Rightarrow$ reject.

* Next, consider $l=4$, $m=3$.
$P_l^m(\cos\theta) \sim \sin^3\theta \cdot \cos\theta$, which has the required odd behavior about $\theta = \pi/2$.

* Hence, $$\underbrace{V(r,\theta,\phi)\Big|_{\text{outside}}}_{\text{leading-order contribution}} \sim V_0\left(\frac{R}{r}\right)^5 \sin^3\theta \cos\theta \sin 3\phi$$

[6].

Q7. $\vec{B}$ inside perfect diamagnet $= 0$



(a) $\Rightarrow$ $B_0 + \frac{2}{3}\mu_0 (K_a + K_b) = 0$

$\Rightarrow$ $\boxed{K_a + K_b = -\frac{3}{2} B_0/\mu_0}$

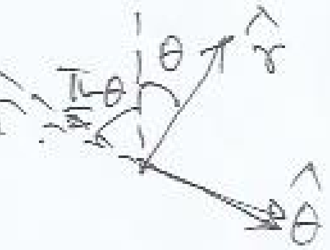
[2]

(b) $\vec{B}$ inside shell $= \hat{z}\left(B_0 + \frac{2}{3}\mu_0 K_b\right) + \vec{B}\Big|_{\text{due to } K_a}$

$\hookrightarrow = \vec{\nabla}\times\vec{A}\Big|_{K_a}$

$\vec{A}\Big|_{\substack{\text{due to}\\ K_a}}$ (outside) $= \frac{\mu_0}{3} K_a \frac{a^3}{r^2} \sin\theta\, \hat{\phi}$.

$\vec{B}\Big|_{\substack{\text{due to}\\ K_a}} = \vec{\nabla}\times\vec{A}$ | $B_\theta\Big|_{\substack{\text{due to}\\ K_a}} = \frac{\mu_0 K_a}{3} \frac{a^3}{r^3} \sin\theta$



$\therefore$ $B_\theta\Big|_{\text{total}} = -\left[B_0 + \frac{2}{3}\mu_0 K_b - \frac{\mu_0 K_a}{3}\frac{a^3}{r^3}\right] \sin\theta$.

[5]

$(\hat{\theta}\times\hat{r} = -\hat{\phi})$

(c) $\underset{\text{(mag.)}}{\vec{K}_b} = K_b \sin\theta\, \hat{\phi} = \vec{M}\times\hat{n} = \vec{M}\times\hat{r} = -M_\theta\, \hat{\phi}\Big|_{r=b}$.

$M_\theta\Big|_{r=b} = B_\theta\Big|_{r=b} \Big/ \mu_0\left(\frac{1}{\chi_m}+1\right)$

Hence, $\boxed{K_b = \dfrac{-B_\theta\big|_{r=b}}{\mu_0\left(\frac{1}{\chi_m}+1\right)}}$

[3]

Q8. (a) For bd. condⁿ (i)

$$V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left\{ \frac{\sin \frac{\pi y}{b}}{\sinh \frac{\pi x}{b}} \right\}$$

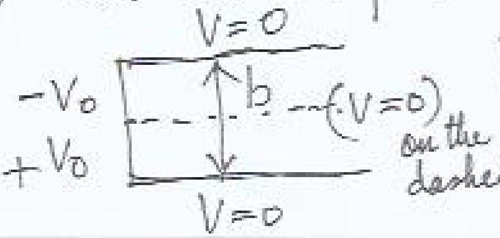
For bd. condⁿ (ii)

$$V(\rho,\phi) = \frac{2V_0}{\pi} \tan^{-1}\left\{ \frac{\sin\phi}{\sinh\xi} \right\} \qquad \left(\text{where, } \frac{\rho}{b} = e^{\xi}\right)$$

$$\left\{ \sinh\xi = \frac{1}{2}\left(e^{\xi} - e^{-\xi}\right) = \frac{1}{2}\left(\frac{\rho}{b} - \frac{b}{\rho}\right) = \frac{1}{2}\left(\frac{\rho^2 - b^2}{2\rho b}\right). \right.$$

The mapping between (ξ, ϕ) and (x, y) is obvious.

(b) For the modified bd. condⁿ:



$$V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left\{ \frac{\sin \pi y/(b/2)}{\sinh \pi x/(b/2)} \right\}$$

$$= \frac{2V_0}{\pi} \tan^{-1}\left\{ \frac{\sin 2\pi y/b}{\sinh 2\pi x/b} \right\}$$

[Multiplication of both x and y terms by the SAME factor required for the Laplace eqⁿ $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$]

Hence, for bd. condⁿ (iii)

$$V(\rho,\phi) = \frac{2V_0}{\pi} \tan^{-1}\left\{ \frac{\sin 3\phi}{\sinh 3\xi} \right\}$$

$$\left\{ \begin{aligned} &\sinh 3\xi \\ &= \left[\left(e^{\xi}\right)^3 - \left(e^{-\xi}\right)^3\right]/2 \\ &= \left[(\rho/b)^3 - (b/\rho)^3\right]/2 \\ &= \frac{\rho^6 - b^6}{2\rho^3 b^3}. \end{aligned} \right.$$